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ON A FORMULA OF INTERPOLATION.

By Dr. E. D. ROE, JR., Associate Professor of Mathematics in Syracuse University.

(Presented at the 1900 December Meeting of the American Mathematical Society.)

§1. INTRODUCTORY STATEMENTS.

Last July the writer obtained an extension of the formula $f(x+n)=(1+\triangle)^n f(x)$ for a positive integral n, of finite differences, to any real value of n and within the limits of convergence of the series, but only for the simplest value of the operator $(1+\triangle)^{\nu/\lambda}$. The other roots of the operator would be found by multiplying the simplest one by the other ν/λth roots of unity. Neither did the writer contemplate an extension to imaginary values of n. That would have to be done by definition. He has recently corresponded with Dr. McClintock and Prof. H. L. Rice, both of whom kindly communicated the fundamentals of their views.

Dr. McClintock has treated the question in this manner: (*American Journal of Mathematics*, 1879, Vol. II, page 117) "I find it better to define E^h as a compound symbol representing that simple operation which changes $(\phi)x$ into $\phi(x+h)$, whatever be the meaning of h. In this light we must regard E, when without an index, as an abbreviated form of E^1." The reason which he gave for his definition was as follows: "Yet if $E\phi(x)=\phi(x+1)$ expresses the definition of E, it will require considerable labor to prove that in all cases $E^h\phi(x)=\phi(x+h)$, and then only when h expresses some positive or negative quantity; and the ar-

gument will not be free from ambiguity, since, for example, it might be hard to prove that $E^{\frac{1}{2}}\,\phi(x)$ cannot be its own opposite, namely, $-\phi(x+\frac{1}{2})$."

Professor Rice writes as follows: "In my work on 'Interpolation' I have proved the truth of the fundamental formula

$$f(x+y)=(1+\triangle)^y f(x)$$

in so far as it *can* be proven. I have put the matter in this form:

(1). I have rigorously shown that the formula is true for *any* value of y *provided* the *differences* become *absolutely constant* at some particular order.

(2). I have further proved that if the differences become *sensibly*, but *not* absolutely constant at some order not too remote, then the formula is applicable, and will give as close an approximation to the truth as the nature and accuracy of the tabular quantities may permit."

His proof is obtained through the medium of Taylor's formula, two different series in terms of finite differences being related to the same Taylor's series, and thus to each other. On the other hand the proof to be given here is rather by the immediate and direct relation of two series in terms of finite differences to each other, and independently of any Taylor's series. It is first proved that the formula $f(x+y)=(1+\triangle)^y f(x)$ holds for a positive commensurable fraction y, then for a negative integer, next for a negative commensurable fraction, and lastly for an incommensurable y.

Whatever the value of m, the following abbreviation is used throughout this paper:

$$\begin{bmatrix} m \\ r \end{bmatrix}=\frac{m(m-1)\dots(m-r+1)}{r!} \qquad \dots\dots\dots\dots\dots (1).$$

§2. The Two Series of the Same Function, and Their Differences.

Two series of the same function are taken:

$$f(x+0)=u_0,\ f(x+1)=u_1,\ \dots\ f(x+r)=u_r,\ \dots \text{ with differences } \triangle u\text{'s}\dots\dots(2).$$

$$f(x+0)=v_0.\ f(x+\frac{1}{\lambda})=v_1,\ \dots f(x+\frac{\nu}{\lambda})=v_\nu,\ \dots \text{ with differences } \delta v\text{'s}\dots(3).$$

Here ν and λ are both integers. Analogously to $u_r=(1+\triangle)^r u_0$, we can also prove,

$$v_\nu=(1+\delta)^\nu\, v_0\dots\dots\dots\dots\dots\dots\dots(4)$$

since we may write

$$v_0=f(x,\ 0),\ v_1=f(x,\ 1),\ \dots\ v_\nu=f(x,\ \nu).$$

Also,

$$v_0=v_1,\ v_\lambda=u_1,\ \dots\ v_{r\lambda}=u_r,\ \dots,$$

and for example,

$$v_1 - v_0 + v_2 - v_1 + \dots . v_\lambda - v_{\lambda-1} = v_\lambda - v_0 = u_1 - u_0$$
$$= \delta v_0 + \delta v_1 + \dots . \delta_{\lambda-1} = \triangle u_0 \dots \dots \dots \dots (5).$$

and similarly for the other Δu's. As the latter are the sum of λ terms of the δv's, we have by the formula for such a sum,

$$\Delta u_0 = \binom{\lambda}{1}\delta v_0 + \binom{\lambda}{2}\delta^2 v_0 + \dots . \binom{\lambda}{\lambda}\delta^\lambda v_0 .$$

$$\Delta u_1 = \binom{\lambda}{1}\delta v_\lambda + \binom{\lambda}{2}\delta^2 v_\lambda + \dots . \binom{\lambda}{\lambda}\delta^\lambda v_\lambda .$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Delta u_r = \binom{\lambda}{1}\delta v_{r\lambda} + \binom{\lambda}{2}\delta^2 v_{r\lambda} + \dots . \binom{\lambda}{\lambda}\delta^\lambda v_{r\lambda} .$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (6).$$

$$\Delta^2 u_0 = \binom{\lambda}{1}(\delta v_\sigma - \delta v_0) + \binom{\lambda}{2}(\delta^2 v_\lambda - \delta^2 v_0) + \dots \binom{\lambda}{\lambda}(\delta^\lambda v_\lambda - \delta^\lambda v_0).$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Delta^{r+1} u_0 = \binom{\lambda}{1}(\delta v_\lambda - \delta v_0)_r + \binom{\lambda}{2}(\delta^2 v_\lambda - \delta^2 v_0)_r + \dots . \binom{\lambda}{\lambda}(\delta^\lambda v_\lambda - \delta^\lambda v_0)_r$$

where $(\delta^\kappa v_\lambda - \delta^\kappa v_0)_r = \delta^\kappa v_{r\lambda} - \binom{r}{1}\delta^\kappa v_{(r-1)\lambda} + \binom{r}{2}\delta^\kappa v_{(r-2)\lambda} - \dots .$

Expressing $v_{r\lambda}$, $v_{(r-1)\lambda}$, ... etc., in terms of v_0,

$$\Delta^{r+1} u_0 = \binom{\lambda}{1}(\Delta^r(0, r)\delta^{r+1} v_0 + \Delta^r(0, r+1)\delta^{r+2} v_0 + \dots .)$$

$$+ \binom{\lambda}{2}(\Delta^r(0, r)\delta^{r+2} v_0 + \dots) + \dots . (7)$$

where

$$(x, H) = \binom{x\lambda}{H}, \ \Delta^r(0, H) = \binom{r\lambda}{H} - \binom{r}{1}\binom{(r-1)\lambda}{H} + \dots . (-1)^{r-1}\binom{r}{r-1}\binom{\lambda}{H},$$

and all the coefficients of the δv's in (7) of lower order than the $(r+1)$st are zero by means of the relation

$$\Delta^r(x-r, H) = 0, \qquad (8)$$

if $r > H$. This is seen by considering the fact that

$$\left[\begin{matrix}(x-r)\lambda\\ H\end{matrix}\right]=\frac{(x-r)\lambda(x-r)\lambda-1)\ldots((x-r)\lambda-H+1)}{H\,!}$$

is of degree H in x; therefore $\varDelta^r(x-r,\,H)$ is of degree $H-r$ in x, and therefore zero, if $r>H$. We may also notice that,

$$\varDelta^r(0,\,r)=A_r\lambda^r=\lambda^r,$$

$$\varDelta^r(0,\,r+1)=A_{r+1}\lambda^r(\lambda-1)=\frac{r}{2}\lambda^r(\lambda-1), \tag{9}$$

where A_r, and A_{r+1} are the coefficients of x^r and x^{r+1} in $(e^x-1)^r$.* From this

$$\varDelta^{r+1}u_0=\lambda^{r+1}\delta^{r+1}v_0+\frac{r}{2}\lambda^{r+1}(\lambda-1)\delta^{r+2}v_0+\ldots. \tag{10}$$

that is, each $\varDelta^\kappa\,u_0$ is expressed in terms of $\lambda^\kappa\,\delta^\kappa\,v_0$, and higher differences than those of order κ in the δv's. We see:

If the κth order of differences in the δv's is constant, $\delta^{\kappa+r}v_0=0$. and $\varDelta^\kappa\,u_0=\lambda^\kappa\,\delta^\kappa\,v_0$ exactly and $\varDelta^{\kappa+r}u_0=0$, and conversely if $\varDelta^{\kappa+r}u_0=0$, it follows that $\delta^{\kappa+r}v_0=0$. and $\varDelta^\kappa\,u_0=\lambda^\kappa\,\delta^\kappa\,v_0$, for a series of linear functions of the δv's beginning with the $(\kappa+1)$st order of differences will be zero independently of λ. Hence the order of constancy is the same for both series.

§3. The Series $(1+\varDelta)^{\nu/\lambda}f(x)$.

The series, $(\nu^1=\frac{\nu}{\lambda})$,

$$u_0+\left[\begin{matrix}\nu^1\\ 1\end{matrix}\right]\varDelta u_0+\left[\begin{matrix}\nu^1\\ 2\end{matrix}\right]\varDelta^2u_0+\ldots\left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\varDelta^ru_0+\ldots \tag{11}$$

within the limits of convergence, is taken as starting point in the proof. By convention, this development may be neatly expressed symbolically as $(1+\varDelta)^{\nu/\lambda}u_0$, if it is agreed to understand that only the simplest expansion of the operator is taken, and not any one of the values it might have from its similarity to a fractional power of a real binomial $1+x$. This however will be mere notation and does not form any necessary part of the proof.

§4. The Convergency of the Series.

There are various tests of convergency of series. One of the most elementary gives the following result: If beginning with and after a value r,

$$\mathrm{mod}\frac{\nu^1-r+1}{r}\,\frac{\varDelta^ru_0}{\varDelta^{r-1}u_0}<\kappa<1, \tag{12}$$

---

*See "Note on Integral and Integro-Geometric Series" by the author, *Annals of Mathematics*, October, 1897, page 184.

the series is convergent. The series is certainly convergent at least :

1. If the $\varkappa$th order of differences of the Δu's is constant; then $\Delta^{\kappa+1}u_0 = \Delta^{\kappa+2}u_0 = \ldots\ldots = 0$, and the series breaks off and is a finite series.

2. If

$$2 > \mathrm{Lim}\frac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

for, since $\quad \dfrac{\Delta^r u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1 - \Delta^{r-1}u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} - 1$, and

$$-1 < \mathrm{Lim}\frac{\Delta^r u_0}{\Delta^{r-1}u_0} < 1,$$

the condition of convergency is satisfied. It is seen that Rice's two cases are contained here, the first in the first, and the second in the second condition. It was not the object here to go into the question of how far the series is convergent, but only to point out the possibility of its convergence. The fuller meaning of the second condition, and the possibility of farther conditions by finer tests might form a subject of future investigation.

§5. The Δu Series Expressed in Terms of the δv's.

Expressing the series $u_0 + \binom{\nu^1}{1}\Delta u_0 + \binom{\nu^1}{2}\Delta^2 u_0 + \ldots\ldots \binom{\nu^1}{r}\Delta^r u_0 + \ldots\ldots$ in terms of the δv's by means of (7), the collected coefficient c_r of $\delta^r v_0$ becomes

$$c_r = \sum_{\kappa=0}^{\kappa=r-1} \binom{\varkappa+1}{\nu^1} \sum_{\mu=1}^{\mu=r-\kappa} \binom{\lambda}{\mu} \Delta^\kappa (0, r-\mu) \ldots\ldots\ldots\ldots \ldots (14).$$

By means of the relations

$$\binom{\rho+\sigma}{r} = \binom{\rho}{0}\binom{\sigma}{r} + \binom{\rho}{1}\binom{\sigma}{r-1} + \ldots\ldots \binom{\rho}{r-1}\binom{\sigma}{1} + \binom{\rho}{r}\binom{\sigma}{0} \ldots\ldots\ldots (15)$$

and $\Delta^\kappa (0, \sigma) = 0$, where $\sigma < \varkappa$, it can be reduced to

$$c_r = \sum_{\kappa=1}^{\kappa=r} \binom{\nu^1}{\varkappa} \Delta^\kappa (0, r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

$\Delta^\kappa (0, r)$ is of the form $a_0\lambda^r + a_1\lambda^{r-1} + \ldots\ldots a_{r-\kappa}\lambda^\kappa$. $\binom{\nu^1}{\varkappa}\Delta^\kappa (0, r)$ is therefore an integral function of λ and ν, of degree $r-1$ in λ, and of degree $\varkappa$ in ν; and c_r is integral and of no higher degree than $r-1$ in λ, and of degree r in ν. We

shall show that it does not contain λ at all. $\binom{\varkappa\lambda}{r}$ is an integral function of $\varkappa\lambda$ of degree r, and is divisible by $\varkappa\lambda$; it is of the form $(\varkappa\lambda)^r+a_1(\varkappa\lambda)^{r-1}+\ldots . a_{r-1}(\varkappa\lambda)$ $=\varkappa^r\lambda^r+\varkappa^{r-1}a_1\lambda^{r-1}+\ldots . \varkappa a_{r-1}\lambda$; $\binom{(\varkappa-1)\lambda}{r}$ is therefore of the form $(\varkappa-1)^r\lambda^r$ $+(\varkappa-1)^{r-1}a_1\lambda^{r-1}+\ldots . (\varkappa-1)a_{r-1}\lambda$. Hence the coefficient of λ^σ in $\Delta^\kappa(0, r)$ is

$$a_{r-\sigma}\frac{\varkappa^\sigma-\varkappa(\varkappa-1)^\sigma+\dfrac{\varkappa(\varkappa-1)}{2!}(\varkappa-2)^\sigma-\ldots . (-1)^{\kappa-1}\varkappa.1^\sigma}{r!}$$

$$=\frac{1}{r(r-1)\ldots . (\sigma+1)}a_{r-\sigma}A_{(\kappa)+\sigma-\kappa}$$

where $A_{(\kappa)+\sigma-\kappa}$ is the coefficient of $x^{\sigma-\kappa}$ in $(e^x-1)^\kappa$, and where this is zero if $\sigma<\varkappa$,* which proves the statement that $\Delta^\kappa(0, r)$ contains the factor λ^κ.

§6. The Coefficient c_r of $\delta^r v_0$.

We have shown that c_r is of the form,

$$c_r=f_r(\nu, r)+f_{r_1}(\nu, r)\lambda+\ldots . f_1(\nu, r)\lambda^{r-1}\ldots\ldots\ldots\ldots(17),$$

the indices denoting the degree in ν; we shall prove that

$$f_\kappa(\nu, r)\equiv 0,\ \varkappa=1, 2, \ldots . r-1, \text{ and } c_r\equiv f_r(\nu, r)\equiv\binom{\nu}{r}\quad\ldots\ldots\ldots(18).$$

The equation $c_r-\binom{\nu}{r}=0$, or

$$f_1(\nu, r)\lambda^{r-1}+f_2(\nu, r)\lambda^{r-2}+\ldots . f_r(\nu, r)-\binom{\nu}{r}=0\quad\ldots\ldots\ldots(19),$$

is satisfied by r values of λ,

$$\nu=\varkappa\lambda,\ \lambda=\frac{\nu}{\varkappa},\ \nu^1=\varkappa,\ \varkappa=1, 2, \ldots . r\ \ldots\ldots\ldots\ldots\ldots(20).$$

We have when $\nu^1=\varkappa$,

*See "Note on Integral and Integro-Geometric Series," l. c.

$$c_r=\binom{\varkappa}{1}\Delta(0,\ r)+\binom{\varkappa}{2}\Delta^2(0,\ r)= \ \ldots.\binom{\varkappa}{\varkappa}\Delta^\kappa(0,\ r)\ \ldots\ldots\ldots(21).$$

The terms beyond that containing $\binom{\nu^1}{\varkappa}=\binom{\varkappa}{\varkappa}$ in c_r all vanish because they contain the factor $(\nu-\lambda\varkappa)=0$. The coefficient of $\binom{\frac{\rho\nu}{\kappa}}{r}$ in c_r is, $(\rho<\varkappa)$,

$$\binom{\varkappa}{\rho}-\binom{\varkappa}{\rho+1}\binom{\rho+1}{1}+\binom{\varkappa}{\rho+2}\binom{\rho+2}{2}-\ldots.(-1)^\sigma\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma}$$
$$+\ \ldots.\ (-1)^{\kappa-\rho}\binom{\varkappa}{\varkappa}\binom{\varkappa}{\varkappa-\rho}\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

The general term of this is $(-1)^\sigma\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma}=(-1)^\sigma\binom{\varkappa}{\rho}\binom{\varkappa-\rho}{\sigma}$ and (22) becomes

$$\binom{\varkappa}{\rho}\sum_{\sigma=0}^{\sigma=\kappa-\rho}(-1)^\sigma\binom{\varkappa-\rho}{\sigma}=\binom{\varkappa}{\rho}(1-1)^{\kappa-\rho}=0\ \ldots\ldots\ldots\ldots(23),$$

and this is true for all values of ρ, from $\rho=1$, to $\rho=\varkappa-1$; only the single value $\binom{\varkappa}{\varkappa}\binom{\nu}{r}=\binom{\nu}{r}$ of c_r is left, and

$$c_r=\begin{bmatrix}\nu\\ r\end{bmatrix},\ \text{for}\ \lambda=\frac{\nu}{\varkappa},\ \varkappa=1,\ 2,\ \ldots.r\ldots\ldots\ldots\ldots\ (24),$$

hence (19) is an identity, and

$$c_r\equiv\begin{bmatrix}\nu\\ r\end{bmatrix}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

§7. The Sum of the Series.

By means of the result of §6, we can express the sum of the series (11), viz:

$$u_0+\begin{bmatrix}\nu^1\\ 1\end{bmatrix}\Delta u_0+\begin{bmatrix}\nu^1\\ 2\end{bmatrix}\Delta^2u_0+\ldots.\begin{bmatrix}\nu^1\\ r\end{bmatrix}\Delta^ru_0+\ldots.=v_0+\begin{bmatrix}\nu\\ 1\end{bmatrix}\delta v_0$$
$$+\ldots.+\ldots.\begin{bmatrix}\nu^1\\ r\end{bmatrix}\delta^rv_0+\ \ldots.\begin{bmatrix}\nu\\ \nu\end{bmatrix}\delta^\nu\,v_0=v_\nu=f(x+\frac{\nu}{\lambda}),\ \text{or}$$

$$f(x+\frac{\nu}{\lambda})=f(x)+\frac{\nu}{\lambda}\Delta f(x)+\frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda}-1\right]}{2\,!}\Delta^2f(x)$$
$$+\ldots.\frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda}-1\right]\ldots.\left[\frac{\nu}{\lambda}-r+1\right]}{r\,!}\Delta^rf(x)+\ldots\ldots\ (26).$$

If we use the symbolic abbreviation before suggested with the understanding that by it we mean only the right member of (26), then we have proved,

$$f(x+\frac{\nu}{\lambda})=(1+\Delta)^{\nu/\lambda}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

§8. The Formula (27) Proved by Symbolic Methods.

Immediately after the writer had found the previous non-symbolic proof that the formula $f(x+y)=(1+\Delta)^y f(x)$ holds for a commensurable fractional y, he discovered by symbolic methods a form of proof which is the opposite extreme of simplicity as compared with the non-symbolic. The two proofs confirm each other. We have:

$$\Delta u_0=((1+\delta)^\lambda-1)v_0.$$
$$\Delta u_1=((1+\delta)^\lambda-1)(1+\delta)^\lambda v_0.$$
$$\Delta u_2=((1+\delta)^\lambda-1)(1+\delta)^{2\lambda} v_0.$$
$$\cdots\cdots\cdots$$
$$\Delta u_r=((1+\delta)^\lambda-1)(1+\delta)^{r\lambda} v_0.$$
$$\cdots\cdots\cdots$$
$$\Delta^2 u_0=((1+\delta)^\lambda-1)^2 v_0.$$
$$\Delta^3 u_0=((1+\delta)^\lambda-1)^3 v_0.$$
$$\cdots\cdots\cdots$$
$$\Delta^r u_0=((1+\delta)^\lambda-1)^r v_0. \qquad (28).$$

Hence by the use of the formulas, (28), and confining the meaning of the symbol $(1+\Delta)^{\nu/\lambda}$ as before,

$$(1+\Delta)^{\nu/r}u_0=u_0+\binom{\nu^1}{1}\Delta u_0+\ \ldots\binom{\nu^1}{r}\Delta^r u_0+\ldots$$

$$=v_0+\binom{\nu^1}{1}((1+\delta)^\lambda-1)v_0+\ \ldots\binom{\nu^1}{r}((1+\delta)^\lambda-1)^r v_0+\ldots$$

$$=(1+(1+\delta)^\lambda-1)^{\nu^1}v_0=(1+\delta)^{\nu^1\lambda}v_0=(1+\delta)^\nu v_0=v_\nu=f(x+\frac{\nu}{\lambda}),$$

or $f(x+\frac{\nu}{\lambda})=(1+\Delta)^{\nu/\lambda}f(x)$.

§9. Extension to Negative Exponents.

Confining the symbols as before, and the series to convergency, and n and r to positive integers, $n>r$,

$$f(x+(n-r))=(1+\Delta)^{n-r}f(x)=f((x+n)-r)\ldots\ldots\ldots(29).$$

Also $(1+\Delta)^{-r}f(x+n)=f(x+n)-r\Delta f(x+n)+\ldots$

$$=(1+\Delta)^n(f(x)-r\Delta f(x)+\ldots)$$
$$=(1+\Delta)^n(1+\Delta)^{-r}f(x)$$
$$=(1+\Delta)^{n-r}f(x) \qquad (30).$$

Comparing (29) and (30) we have

$$f((x+n)-r)=(1+\Delta)^{-r}f(x+n)\ldots\ldots\ldots\ldots\ldots\ldots(31),$$

or denoting $(x+n)$ by a single letter,

$$f(x-r)=(1+\Delta)^{-r}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(32).$$

In order to extend to a negative fractional commensurable index, we choose, $-\nu=\varkappa\lambda$, $\varkappa=1, 2, \ldots .r$, in §6, and come out with $c_r=\binom{-\nu}{r}$ always, where we are supposed to have found the value of the simplest series $(1+\Delta)^{-\nu/\lambda}f(x)$ in terms of δv's, and thus we have

$$(1+\Delta)^{-\nu/\lambda}f(x)=(1+\delta)^{-\nu}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Or as in §8, by symbolic operations we arrive at

$$(1+\Delta)^{-\nu^1}u_0=(1+\delta)^{-\nu^1\lambda}v_0=(1+\delta)^{-\nu}v_0.$$

By reasoning analogous to that by which (32) was found,

$$(1+\delta)^{-\nu}f(x)=f(x-\frac{\nu}{\lambda})\ \ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Comparing (33) and (34),

$$f(x-\frac{\nu}{\lambda})=(1+\Delta)^{-\nu/\lambda}f(x)\ \ldots\ldots\ldots\ldots\ldots\ldots(35).$$

§10. Extension to an Incommensurable y.

Here we have to use limits. Let y be incommensurable, y^1 commensurable, and let $y^1\doteq y$. Then by a combination of the previous paragraphs,

$$f(x+y^1)=(1+\Delta)^{y^1}f(x), \text{ and}$$

$$\lim_{y^1\doteq y}f(x+y)^1)=\lim_{y^1\doteq y}(1+\Delta)^{y^1}f(x), \text{ or}$$

$$f(x+y)=(1+\Delta)^{y}f(x)\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Hence within the limits of convergence, and for the confined meaning of the symbol $(1+\Delta)^y$, we have proved formula (36) to hold for all real values whatsoever of y.

Syracuse University, 1 December, 1900.

TWO HYDRAULIC METHODS TO EXTRACT THE nth ROOT OF ANY NUMBER.

By DR. ARNOLD EMCH, University of Colorado.

1. The first method is based upon the construction of a vessel which is bounded by such a surface of revolution that the weight of the water displaced by the surface is always equal to the nth power of the depth of immersion. Suppose that $x=f(y)$ is the equation of a meridian of the surface, Fig. I, for which OY is the axis of revolution, and that the meridian touches the line OX at O. Using one foot as the unit of measure and designating by w the weight of a cubic foot of water which may be assumed as 62.5 pounds, the weight of the displaced water, when the depth of immersion is y, is

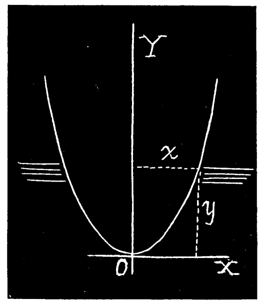

Fig. 1.

$$W=\pi.w\int_0^y x^2.dy=\pi.w\int_0^y f^2(y)dy \;\ldots.(1),$$

and this shall equal y^n. Hence

$$\pi w\int_0^y f^2(y)dy=y^n\ldots.(2),$$

from which $x=f(y)$ is easily found:

$$x=f(y)=\sqrt{\frac{n}{\pi.w}y^{\frac{1}{2}(n-1)}}\;\ldots.(3),\quad\text{or}\quad y=\sqrt{\frac{\pi.w}{n}x^{2/(n-1)}}\ldots.(4).$$

Giving n successively the values 1, 2, 3, ... we find for the equations of the meridians

$$x=\sqrt{\frac{1}{\pi.w}}=0.05052\ldots.(5),$$

$$y=\sqrt{\frac{\pi.w}{2}}x^2=9.9083.x^2\ldots.(6),$$

$$y=\sqrt{\frac{\pi.w}{3}}x=8.0901.x\ldots.(7),$$

etc., (unit=1 foot)

so that for the first, square, and cube roots, the surfaces are respectively a cylin-

der, a paraboloid and a cone of rotation. Now by the principle of Archimedes the weight of the water displaced is equal to the weight of the vessel. Hence, if this weight is Q (in pounds) and if we want to extract the nth root of a given number N, we add to Q a weight P in such a manner that $Q+P=N$. After the vessel is in equilibrium, there is

$$Q+P=N=W=y^n$$

and consequently,

$$y=\sqrt[n]{N}\ldots.(8).$$

The depth of immersion, y, is therefore the nth root of the number N.

In Fig. 2 an apparatus for the extraction of the square root has been sketched. The shape of the paraboloid of rotation is determined by equation (6) and corresponds to the wetted surface of the vessel. Suppose P is the weight to be added at A to make $Q+P=N$, then the difference of the water-levels before and after immersion will be equal to y or the square root of N. The difference in levels may be measured by the hook-guage G. In the mechanical execution of this apparatus it is not difficult to secure any reasonable precision and to eliminate errors of construction and observation.

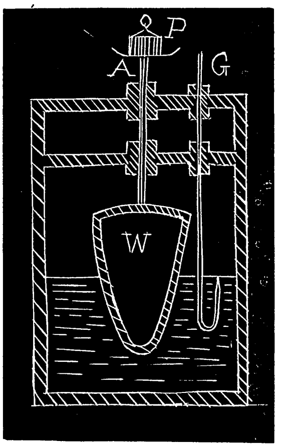

Fig. 2.

2. A second method to extract the nth root of any number may be based upon the problem to find the time necessary of emptying a vessel of given form through a small orifice at the bottom. We assume again a surface of revolution and refer it to the same axes as in the previous problem. Then the time dt to lower the level of the water in the vessel by dy units is expressed by the formula,

$$dt=\frac{\pi.x^2.dy}{a\sqrt{(2gy)}}=\frac{\pi f^2(y)dy}{a\sqrt{(2gy)}}*\ldots.(9),$$

where a is the cross-section of the orifice.

Hence the time of emptying the vessel from the original level y is

$$t=\frac{\pi}{a\sqrt{(2g)}}\int_y^0\frac{f^2(y)dy}{\sqrt{y}}\ldots\ (10).$$

We shall now determine $f(y)$ in such a manner that

$$t=\sqrt[n]{y}=\frac{\pi}{a\sqrt{(2g)}}\int_y^0\frac{f^2(y)dy}{\sqrt{y}}\ldots.(11).$$

*See Merriman's Hydraulics, page 56.

From (11) we find for $x=f(y)$ the expression

$$x=\sqrt{\frac{a\sqrt{(2g)}}{n.\pi}}.y^{(2-n)/4n}\ldots.(12),$$

which for $n=1$, 2, 3, ... *i. e.*, for the first, square, and cube-root, becomes

$$x=\sqrt{\frac{a\sqrt{(2g)}}{\pi}}.y^{\frac{1}{4}}\ \ldots.(13),$$

$$x=\sqrt{\frac{a\sqrt{(2g)}}{2\pi}}\ldots.(14),$$

$$x=\sqrt{\frac{a\sqrt{(2g)}}{2\pi}}.y^{-\frac{1}{12}}\ldots.(15),$$

respectively. The shape of the vessel in case of square-roots is therefore cylindrical, and if the radius of the cylinder is made according to (14) *the time of emptying the cylindrical vessel will be equal to the square root of the original depth of water.*

The physical conditions of the problem make it clear that the first method is more accurate. The determination of the time in the second method is liable to be affected with an error of a higher order than those occuring in the statical extraction of a root.

In the near future the author shall publish a remarkably simple extension of the first method to the solution of an equation of the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots.+a_nx^n=0.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

133. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.**

In Wentworth's Arithmetic he gives a formula $\frac{21}{40}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The number of board feet in a log d inches in diameter and l inches in length is $n=\dfrac{\frac{1}{4}\pi d^2 l}{144}$.

If l is in feet, $n=\dfrac{\frac{1}{4}\pi d^2 l}{12}$.

Substituting $\frac{22}{7}$ for π and 10 for l we have, $n=\frac{5}{6}.\frac{11}{14}d^2$.

Allowing $\frac{1}{5}$ for saw cut, $n=\frac{4}{5}.\frac{5}{6}.\frac{11}{14}d^2=\frac{11}{21}d^2$.

Allowing $\frac{1}{2}$ inch for bark, $n=\frac{11}{21}(d-1)^2$.

For $d=22$, an average value, $(d-1)^2=\frac{441}{440}(d^2-2d)$.

$\therefore\ n=\frac{11}{21}.\frac{441}{440}(d^2-2d)=\frac{21}{40}(d^2-2d)$.

I would propose the formula $n=d^2$ (or $n=(d-1)^2$) for a log 20 feet long, since it is as accurate and much more simple than Wentworth's.

The most accurate formula, however, must be based on the end diameters of the log.

Let d and D represent those diameters.

Board feet in total volume of log 20 feet long

$$=\tfrac{1}{4}\pi.\tfrac{5}{3}\times\frac{d^2+dD+D^2}{3}=\tfrac{5}{9}\times\tfrac{11}{14}(d^2+dD+D^2).$$

(See Philbrick's *Engineer's Manual*, table 23).

Since $(D-d)^2=D^2-2dD+d^2>0$, $D^2+dD+d^2>3dD$.

Hence volume$>\frac{5}{3}\times\frac{11}{14}dD$.

Allowing $\frac{1}{5}$ for saw cut, $n>\frac{4}{3}\times\frac{11}{14}dD=\frac{22}{21}dD$.

Allowing $\frac{1}{22}$ of the above for bark we still have $n>dD$, or, say, $n=dD$ ....(1).

It is easily shown that volume$>\frac{5}{3}\times\frac{11}{14}\left[\dfrac{d+D}{2}\right]^2$

and as before that $n=\left[\dfrac{d+D}{2}\right]^2$....(2).

The author's experience leads him to believe that the above formulas are quite accurate, but that logs will cut a little more into large timbers than the formulas give.

If thought to give too large a result, in extreme cases, we might suggest the formula, $n=d(D-2)$....(3), or $n=\left[\dfrac{d+D}{2}-1\right]^2$....(4).

At all events the forms suggested should be used.

ALGEBRA.

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111. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equation $x(y+z)=a(x+y+z)$, $y(x+z)=b(x+y+z)$, $z(x+y)=c(x+y+z)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Let $x+y+z=s$. Then (1)+(2)+(3) gives $xy+xz+yz=\frac{1}{2}(a+b+c)s$..(4).
(4)−(1) gives $yz=\frac{1}{2}(b+c-a)s$....(5).
(4)−(2) gives $xz=\frac{1}{2}(a+c-b)s$....(6).
(4)−(3) gives $xy=\frac{1}{2}(a+b-c)s$....(7).
(6)÷(7) gives $z/y=(a+c-b)/(a+b-c)$....(8).

(8) multiplied by (5) gives $z=\sqrt{\dfrac{(a+c-b)(b+c-a)s}{2(a+b-c)}}$....(9).

Similarly, $y=\sqrt{\dfrac{(a+b-c)(b+c-a)s}{2(a+c-b)}}$, $x=\sqrt{\dfrac{(a+b-c)(a+c-b)s}{2(b+c-a)}}$....(10, 11).

(9)+(10)+(11) gives $\sqrt{s}=\dfrac{2ab+2ac+2bc-a^2-b^2-c^2}{\sqrt{[2(a+b-c)(a+c-b)(b+c-a)]}}$....(12).

(12) in (9), (10), (11) gives

$$x(b+c-a)=y(a+c-b)=z(a+b-z)=\tfrac{1}{2}(2ab+2ac+2bc-a^2-b^2-c^2).$$

Also solved by *J. M. BOORMAN*, *W. H. CARTER*, *C. C. CROSS*, *LESLIE L. LOCKE*, *COOPER D. SCHMITT*, *ELMER SCHUYLER*, *J. SCHEFFER*, *B. F. YANNEY*, *J. W. YOUNG*, and *M. A. GRUBER*.

GEOMETRY.

138. Proposed by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

K is the middle point of any chord AB of a given circle. CD and EF are any two chords passing through K. CF and ED intersect AB at M and N, respectively. Prove that KM equals KN.

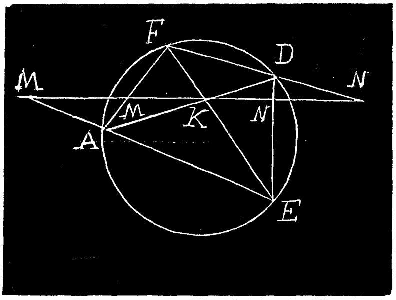
I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; P. S. BERG, B. Sc., Larimore, N. D.; and F. B. FILLMAN, Chester, Pa.

CASE I. N and M within the circle.
$\triangle DKN/\triangle CKM=DK.NK/CK.MK$....(1).
$\triangle EKN/\triangle FKM=EK.NK/FK.MK$....(2).
$\triangle CKM/\triangle EKN=CM.CK/EN.EK$....(3).
Multiplying (1) by (3),

$$\frac{\triangle DKN}{\triangle EKN}=\frac{DK.NK.CM}{MK.EN.EK}=\frac{DN}{EN}....(4).$$

Multiplying (2) by (3),

$$\frac{\triangle CKM}{\triangle FKM}=\frac{NK.CM.CK}{EN.FK.MK}=\frac{CM}{FM}....(5).$$



From (4) and (5),
$NK/MK = DN.EK/DK.CM \ldots (6)$.
$NK/MK = EN.FK/CK.FM \ldots (7)$.
Multiplying (6) by (7),

$$\therefore \frac{NK^2}{MK^2} = \frac{DN.EN.EK.FK}{CM.FM.DK.CK} \ldots (8).$$

But $EK.FK = DK.CK = AK^2$.

$$DN.EN = BN.AN = (BK - KN)(BK + KN) = BK^2 - KN^2.$$

$$CM.FM = AM.MB = (BK - MK)(BK + MK) = BK^2 - KM^2.$$

$$\therefore \frac{NK^2}{MK^2} = \frac{BK^2 - NK^2}{BK^2 - MK^2}.$$

$\therefore NK^2.BK^2 - NK^2.MK^2 = MK^2.BK^2 - NK^2.MK^2$.
$\therefore NK^2.BK^2 = MK^2.BK^2$.
$\therefore NK = MK$.

Case II. N and M without the circle.
$\triangle DKN / \triangle CKM = DK.NK/CK.MK \ldots (9)$.
$\triangle FKN / \triangle EKM = FK.NK/EK.MK \ldots (10)$.
$\angle KDN$ is the supplement of $\angle FDK$ and therefore the supplement of $\angle KEC$.
$\therefore \triangle EKM / \triangle DKN = KE.ME/KD.ND \ldots (11)$.
Multiplying (9) by (11),

$$\frac{\triangle EKM}{\triangle CKM} = \frac{NK.KE.ME}{ND.CK.MK} = \frac{ME}{MC} \ldots (12).$$

Multiplying (10) by (11),

$$\frac{\triangle FKN}{\triangle DKN} = \frac{FK.NK\ ME}{KD.ND.MK} = \frac{NF}{ND} \ldots (13).$$

From (12) and (13).
$NK/MK = ND.CK/MC.KE \ldots (14)$.
$NK/MK = NF.KD/FK.ME \ldots (15)$.
Multiplying (14) by (15),

$$\frac{NK^2}{MK^2} = \frac{ND.NF.\ CK.KD}{MC.ME.FK.KE} = \frac{ND.NF}{MC.ME} = \frac{NB.NA}{MB.MA} = \frac{(NK - BK)(NK + BK)}{(MK - BK)(MK + BK)}$$

$$=\frac{NK^2-BK^2}{MK^2-BK^2}.$$

$\therefore\ NK^2.MK^2-NK^2.BK^2=NK^2.MK^2-MK^2.BK^2$.

$\therefore\ NK^2.BK^2=MK^2.BK^2$.

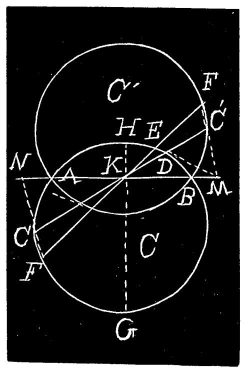
$\therefore\ NK=MK$.

II. Solution by F. E. MILLER, A. M., Professor of Mathematics, Otterbein University, Westville, Ohio, and P. C. CULLEN, Indianola, Neb.

K the mid-point of chord AB, and CD and EF chords through K.

To prove that the joins CF and ED meet AB equidistant from K.

Through A and B draw circles $C'\equiv C$ and produce CD and EF to C' and F'.



From symmetry we see that FC and $F'C'$ meet AB equidistant from K and are parallel.

$\angle EDC=\angle EFC=\angle EF'C'$ and hence $EDC'F'$ are concyclic. Then is $F'C'$ and ED meet in M, $MD.ME=$ $=MC'.MF'$, or the tangents from M to the circles C and C' are equal and therefore M is on the radical axis, *i. e.* on AB.

$\therefore\ NK=MK$.

Again by projection.

Project circle C on a plane through AB so that the projection of HG perpendicular to AB may have the projection of K as its mid-point. Then the circle becomes an ellipse with K as center and CD and EF as diameters, and ED and FC meeting the major axis AB equally distant from the center K. But points on AB are not changed by the projection. Therefore N and M are always equidistant from K.

A second solution by Analytical Geometry was furnished by Professor Zerr.

CALCULUS.

104. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the differential equation corresponding to $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=[a(x-y)]$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

$\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)\ \ldots\ldots(1)$.

$xdx/\sqrt{(1-x^2)}+ydy/\sqrt{(1-y^2)}=a(dy-dx)\ \ldots\ldots(2)$.

Eliminating a between (1) and (2) and reducing we get

$$\{xy-1-\sqrt{[(1-x^2)(1-y^2)]}\}\sqrt{[1-y^2]}dx$$
$$=\{xy-1-\sqrt{[(1-x^2)(1-y^2)]}\}\sqrt{[1-x^2]}dy.$$

$\therefore\ dx/dy=\sqrt{(1-x^2)}/\sqrt{(1-y^2)}$.

NOTE ON RADIUS OF CURVATURE.

By **GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.**

The following method of deducing the formula for the radius of curvature seems to have some pedagogical advantages. The ordinary method where infinitesimals are used leaves some doubt in the mind of the beginner as to the exactness of the result. Especially is this true when the center of curvature is defined as the intersection of two consecutive normals.

Let (x_1, y_1), (x_2, y_2) be any two points of the curve, m_1, m_2 the slope of the tangents at these points. Then

$$x+m_1y=x_1+m_1y_1,$$
$$x+m_2y=x_2+m_2y_2,$$

are the equations of the normals. Solving these equations we find that the normals intersect at the point whose coördinates are

$$x=\frac{m_2x_1+m_1m_2y_1-m_1x_2-m_1m_2y_2}{m_2-m_1}=\frac{(m_1-m_2)x_2-m_2(x_1-x_2)-m_1m_2(y_1-y_2)}{m_1-m_2}$$

$$y=\frac{(x_1-x_2)+(m_1-m_2)y_1+m_2(y_1-y_2)}{m_1-m_2};$$

$$\text{or,}\quad x=x_2-\frac{m_2}{\dfrac{m_1-m_2}{x_1-x_2}}-\frac{m_1m_2\left(\dfrac{y_1-y_2}{x_1-x_2}\right)}{\dfrac{m_1-m_2}{x_1-x_2}},$$

$$y=y_1+\frac{1+m_2\dfrac{y_1-y_2}{x_1-x_2}}{\dfrac{m_1-m_2}{x_1-x_2}}.$$

When (x_2, y_2) approaches (x_1, y_1), we have

$$\lim(m_2)=m_1=\frac{dy_1}{dx_1};\quad \lim\left(\frac{y_1-y_2}{x_1-x_2}\right)=\frac{dy_1}{dx_1};\quad \lim\left(\frac{m_1-m_2}{x_1-x_2}\right)=\frac{d^2y_1}{dx_1^2}.$$

$$\text{Then}\quad x-x_1=-\frac{dy_1}{dx_1}\left\{\frac{1+\left(\dfrac{dy_1}{dx_1}\right)^2}{\dfrac{d^2y_1}{dx_1^2}}\right\},\quad y-y_1=\frac{1+\left(\dfrac{dy_1}{dx_1}\right)^2}{\dfrac{d^2y_1}{dx_1^2}}.$$

The distance of point of intersection from foot of normal is therefore

$$\rho=\frac{\left[1+\left(\frac{dy_1}{dx_1}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y_1}{dx_1^2}}.$$

The equation of a circle whose center is the point of intersection and which passes through the foot of the normal is

$$(x-x_1)^2+(y-y_1)^2=\rho^2.$$

Differentiating this twice and eliminating x, y, we find

$$\rho=\frac{\left[1+\left(\frac{dy_1}{dx_1}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y_1}{dx_1^2}}.$$

This shows that a circle having the same slope and same value of $\frac{d^2y}{dx^2}$ at its point of intersection with a given curve has its center at the limiting point of intersection of two normals.

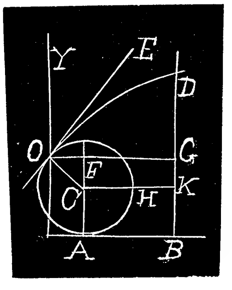
MECHANICS.

100. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University; Cambridge, Mass.

A man, riding a bicycle, runs through a puddle of water and a bit of mud is thrown from the rear wheel and alights on the crown of his hat. Supposing the wheel 28 inches in diameter, that the man's head is 6 feet above ground, that the saddle is 1 foot in front of the rear wheel, and that the mud left the wheel at a point 30° from highest point of wheel, how long will it take a man to ride a mile at this rate?

Solution by **G. B. M. ZERR**, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and **L. R. INGERSOLL**, Student Colorado College, Colorado Springs, Col.

As the particle of mud and man have the same uniform velocity forward, it is not necessary to consider such motion. The result will be the same if we regard the man at rest and the hind wheel of the bicycle revolving with the same velocity as the man is moving forward. Let O be the origin of the coördinates, D the top of the man's head, $\angle EOG=\angle OCF=\theta$. OE the tangent to the wheel at O, $CO=a=14$ inches, $HK=12$ inches, $BD=72$ inches, $g=32.16$ feet. Then $y=x\tan\theta-gx^2/2v^2\cos^2\theta$, is the equation to OD, the path of the mud.



$$\therefore\ v=\frac{x}{2\cos\theta}\sqrt{\frac{2g}{x\tan\theta-y}}\ \ldots.(1).$$

$x=OG=OF+FG=OF+CH+HK=26+14\sin\theta.$

$y=GD=BD-BK-KG=72-14-14\cos\theta=58-14\cos\theta.$

$$\therefore\ v=\frac{(13+7\sin\theta)}{\cos\theta}\sqrt{\frac{g}{(13+7\sin\theta)\tan\theta+7\cos\theta-29}}.$$

When $\theta=30°$, $v=59\sqrt{-3}$ inches, an impossible result.

$\therefore$ $GD>$ than the intersection made by the particle on BD and indicates that the mud would never get 6 feet above the ground.

Let $\theta=60°$, $v=273.17$ inches$=22.76$ feet per second.

$t=5280\div22.76=231.98$ seconds$=3$ minutes, 51.98 seconds, time required to ride a mile.

101. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the center of gravity of a cone that has a specific gravity of 1 (one) at the top and 2 (two) at the base.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, The Temple College, Philadelphia, Pa.; **WILLIAM W. LANDIS, A. M.**, Dickinson College, Carlisle, Pa.; **H. C. WHITAKER, Ph. D.**, Manual Training School, Philadelphia, Pa.

Let $y=m(x-a)$ be the equation to the generator of the cone.

$$\text{Then } \bar{x}=\frac{\int\rho y^2 x dx}{\int\rho y^2 dx}=\frac{\int_a^{2a}\rho x(x-a)^2 dx}{\int_a^{2a}\rho(x-a)^2 dx}.$$

By the conditions of the problem, $\rho=x/a$.

$$\therefore\ \bar{x}=\frac{\int_a^{2a}x^2(x-a)^2 dx}{\int_a^{2a}x(x-a)^2 dx}=\frac{\frac{31a^5}{30}}{\frac{7a^4}{12}}=\frac{62a}{35}.$$

$\bar{y}=0.$ $\dfrac{62a}{35}-a=\dfrac{27a}{35}=$the distance of the center of gravity from the vertex.

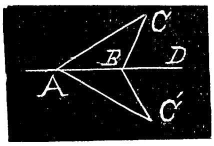
AVERAGE AND PROBABILITY.

70. Proposed by Professor MILLER.

A ship at A observes another at B, whose course is unknown. Supposing their speed the same, prove that the chance of their coming within a given distance, d, of each other is always $(2/\pi)\sin^{-1}(d/a)$, whatever the course taken by A; provided its inclination to AB is not greater than $\cos^{-1}(d/a)$, where $AB=a$. [From *Cambridge Mathematical Tripos*, 1871.]

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $AB=a$, $\angle CAB$ which ship A makes with $AB=\theta$, $\angle CBD$ which ship B makes with $AB=\varphi$ where C is the intersection of the two courses. Then $\angle ACB=(\varphi-\theta)$.



Let v=velocity of each ship, $AC=b$, $BC=c$.

Then in time t, the ship is distant from C, $b-vt$. B is distant from C, $c-vt$.

$\therefore\ d^2=(b-vt)^2+(c-vt)^2-2(b-vt)(c-vt)\cos(\varphi-\theta)$ ....(1).

Differentiating with reference to t for a minimum, we get

$$v(b-vt)+v(c-vt)=[v(c-vt)+v(b-vt)]\cos(\varphi-\theta).$$

$\therefore\ t=(b+c)/2v$....(2).

Substituting (2) in (1) we get $d=(b-c)\cos\frac{1}{2}(\varphi-\theta)$.

But $b=a\sin\varphi/\sin(\varphi-\theta)$, $c=a\sin\theta/\sin(\varphi-\theta)$.

$\therefore\ d=\dfrac{a\cos\frac{1}{2}(\varphi-\theta)(\sin\varphi-\sin\theta)}{\sin(\varphi-\theta)}$.

Now $\sin\varphi-\sin\theta=2\cos\frac{1}{2}(\varphi+\theta)\sin\frac{1}{2}(\varphi-\theta)$.

$\therefore\ d=a\cos\frac{1}{2}(\varphi+\theta)$.

$\therefore\ \varphi=2\cos^{-1}(d/a)-\theta=\theta_1$.

Let $\cos^{-1}(d/a)=\beta$.

$$\therefore\ \text{Chance}=\frac{\int_{-\beta}^{\beta}\int_{\theta_1}^{2\pi-\theta_1}d\theta d\varphi}{\int_{-\beta}^{\beta}\int_{0}^{2\pi}d\theta d\varphi}=\frac{\int_{-\beta}^{\beta}(2\pi-2\theta_1)d\theta}{4\pi\beta}=\frac{(\pi-2\beta)}{\pi}.$$

$\therefore\ p=(2/\pi)[\frac{1}{2}\pi-\cos^{-1}(d/a)]=(2/\pi)\sin^{-1}(d/a)$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

138. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

"A pound of gold may be drawn into a wire that would extend around the earth." What would be the diameter of such a wire if the specific gravity of gold is 19.36 and the distance is 24,900 miles?

139. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

The *ratio* of the interest to the true discount on a certain principal for a certain time at a certain rate per cent. per annum, is $m=21$ to $n=20$. What is the rate per cent.?

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

ALGEBRA.

127. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Sum to n terms the series

$$\frac{4}{1.2.3}\cdot\frac{1}{3}+\frac{5}{2.3.4}\cdot\frac{1}{3^2}+\frac{6}{3.4.5}\cdot\frac{1}{3^2}+\ldots.$$

128. Proposed by ELMER SCHUYLER, B. Sc., Teacher of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(1+x^3)(1+x^2)(1+x)=30x^3$.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Prove $f(x)=e^{e^x}=e+\sum_{r=1}^{r=\infty} a_r x^r$, where $a^r=\frac{1}{r\,!}\sum_{\kappa=1}^{\kappa=\infty}\frac{\kappa^r}{\kappa\,!}$.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

GEOMETRY.

157. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the locus of the center of a circle touching a given line and always passing through a given point.

158. Proposed by JOHN MACNIE, A. M., Professor of Latin, University of North Dakota.

Show by a simple diagram that:

(*a*) If the angle-sum of an equilateral triangle is constant, that constant is a straight angle.

(*b*) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(*c*) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

159. Proposed by FRANCIS W. HANAWALT, Professor of Mathematics and Astronomy, Iowa Wesleyan University, Mt. Pleasant, Iowa.

A man desires to lay out a half mile race course by using two circles of 150 feet radius and their internal tangents. How far apart shall the circles be placed?

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

CALCULUS.

119. Proposed by JOHN M. COLAW. A. M., Monterey, Va.

Rectify the Folium of Descartes, the equation of which is $x^3+y^3+3axy=0$.

120. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The axis of a paraboloid of revolution coincides with the generating line of a cylinder; the diameter of the cylinder and the latus-rectum of the parabola are each equal to the common altitude, a. Find the surface and volume of each part into which the paraboloid is divided by the cylinder.

121. Proposed by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.

Solve the differential equation $\left[\frac{d}{dx}+b\right]^n y=\cos ax$.

122. Proposed by B. F. FINKEL. A. M., M. Sc., Professor of Mathematics and Physics. Drury College, Springfield, Mo.

Solve the differential equation $(y-x)\sqrt{(1+x^2)}\frac{dy}{dx}=n(1+y^2)^{\frac{3}{2}}$.

123. Prize Problem. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{4}\pi}\tan\phi d\phi$.

A year's subscription to the MONTHLY will be given to the person sending to the Proposer the first solution of this problem.

*** Solutions of these problems should be sent to J. M. Colaw not later thah March 10.

MECHANICS.

107. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A rough uniform rod, length $2a$, is placed with a length $c(>a)$ projecting over the edge of the table. Prove that the rod will begin to slide over the edge when it has turned through an angle $\tan^{-1}\left[\frac{\mu a^2}{a^2+9(c-a)^2}\right]$.

[From Loudon's *Rigid Dynamics*.]

108. Proposed by **F. P. MATZ, M. Sc., Ph. D.,** Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Prove that the *inclination* of a perfectly rough inclined plane must be $\theta=\sin^{-1}[e^2/(2-e^2)]$, in order that an ellipse of minimum eccentricity e may be capable of resting in equilibrium on the plane.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than March 10,

AVERAGE AND PROBABILITY.

99. Proposed by **E. B. SEITZ.**

A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects in a given diameter. [No. 135, *The Analyst.*]

100. Proposed by **L. C. WALKER,** Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Required the average distance between two points in opposite sides of a regular $2n$—gon.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than March 10.

MISCELLANEOUS.

100. Proposed by **F. P. MATZ, M. Sc., Ph. D.,** Professor of Mathematies and Astronomy, Irving College, Mechanicsburg, Pa.

Determine the maximum value of $(\varphi-\varphi')$, if given electric currents C and C' produce deflections φ and φ' in a tangent galvanometer, so that $\tan\varphi/\tan\varphi'=C/C'$.

101. Proposed by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

A wire is laid along the surface of a right cone semi-vertical angle β so that it cuts the generators everywhere at a constant angle θ. Find the radius of curvature and radius of torsion.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than March 10.

EDITORIAL NOTE.

The MONTHLY begins the twentieth century with eighth volume. Seven volumes are already completed, and we trust that by the coöperation of its friends it may complete many more volumes.

It is desirable to publish more papers in the future, but only such as are of real value and merit. It is not desirable to publish articles which are mere

modifications of subjects treated in the ordinary text-book, but translations of important subjects from the best European writers would be of real value to many of our readers. For example, a translation of Abel's, or Wantzel's Demonstration of the Impossibility of Resolving Algebraically General Equations of a Degree Higher than the Fourth, Hermite's proof that π is transcendental, subjects that are of great interest to teachers of mathematics. Careful translations of such and other subjects will be welcomed both by readers and the editors.

Through the kindness of a very warm friend of the MONTHLY, the expense of its publication for the past year was nearly all met without drawing painfully on the editors, and we trust that all our subscribers will remain with us and support us in our effort to make the MONTHLY indispensable to all teachers of mathematics.

BOOKS.

The Elements of Analytic Geometry. By Albert L. Candy, Ph. D., Adjunct Professor of Mathematics in the University of Nebraska. 8vo. Cloth, 303 pages. Published by the Author, Lincoln, Neb.

This book has many commendable features, among which are to be noticed: The correlation of subjects; the graphic treatment of the Theory of Equations connects it with Algebra; while Chapter V and the sections on Quadrature, Maxima and Minima, introduce to the student the fundamental notions of the Calculus. Numerous exercises in the earlier chapters involving trigonometric work will refresh the student's mind on that subject. The Theory of Parameter Coordinates is briefly but clearly presented. Many historical notes are inserted. In these ways, the book marks a departure from the ordinary texts on the subject. B. F. F.

Field Manual for Engineers. By Philetеus H. Philbrick, C. E., M. S., Member American Mathematical Society, Member American Society of Civil Engineers, Chief Engineer K. C. W. & G. R. R.

The aim of this book the author says, is, 1. To present the subject of the text in a mathematical and logical order. 2. To classify all problems presented so as to be easily referred to. 3. To express the resulting formula of every problem in the form requiring the least numerical computation. 4. To furnish a large number of useful tables, more complete, more extended, and, when possible, with more elementary and appropriate argument than other similar tables. 5. To treat the general problems of railway engineering more extensively than other similar works have done. After having examined the work we believe that the author has accomplished what he aimed to do. He has written a work which is up to the demands made by the march of modern railway systems of the past twenty years. B. F. F.

ERRATA.

In Problem 153, Geometry, page 234, second line, for "the point" read *a focus*.

Page 299, line 12 from top, in denominator of $\tan^{-1}$ insert $-c^2$.

Page 302, in the notice of Prof. Nichols' Calculus, the publishers should be D. C. Heath & Co., instead of Allyn & Bacon.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. FEBRUARY, 1901. No. 2.

BIOGRAPHY.

KARL FREDERICH GAUSS.

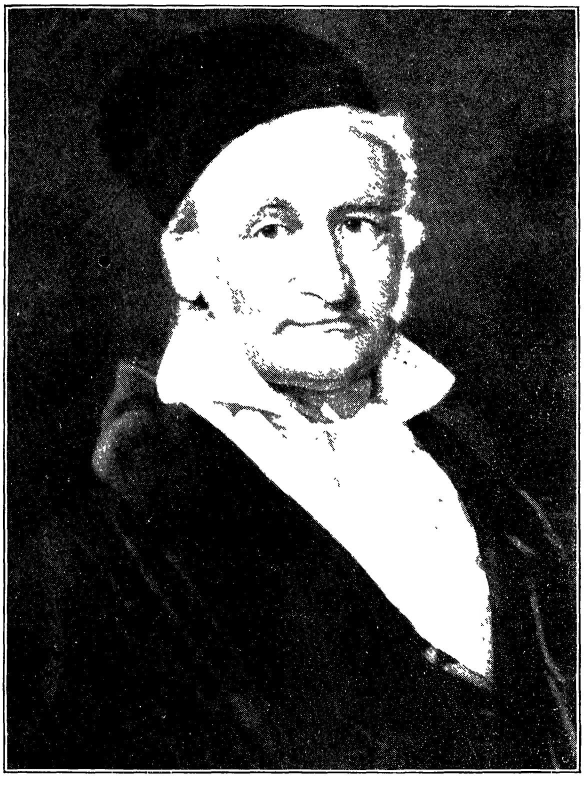
BY B. F. FINKEL.

This versatile and prolific mathematician, Karl Frederich Gauss, was born at Brunswick, Germany, April 30*, 1777, and died at Göttingen on February 23, 1855. His father was a brick-layer and was desirous of profiting by the wages of his son as a laborer, but young Gauss's talents attracted the attention of Bartels, afterwards professor of mathematics at Dorpat, who brought him to the notice of Charles William, Duke of Brunswick. The duke undertook to educate the boy and sent him to the Caroline college, in 1792. By 1795 it was admitted alike by professors and pupils that he knew all that the professors could teach him- It was while at this school that he investigated the method of least squares, and proved by induction the Law of Quadratic Reciprocity. He gave the first rigorous proof of this law and succeeded in discovering eight different demonstrations of it.† While at Caroline college, Gauss manifested as great an aptitude for language as for mathematics, a very general characteristic of eminent mathematicians.

In 1795 Gauss went to Göttingen, as yet undecided whether to pursue philology or mathematics. While at Göttingen he studied mathematics under

*Cf. *Britannica Encyclopedia* and *Century Dictionary*.

†For Gauss's third proof as modified by Dirichlet, see Mathews's *Theory of Numbers*, pages 38-41.



KARL FREDERICH GAUSS.

Abraham Gotthelf Kästner, who was not a very inspiring teacher and who is now chiefly remembered for his *History of Mathematics*, 1796, and by the fact that he was a teacher of the illustrious Gauss. In 1796 he discovered a method of inscribing in a circle a polygon of seventeen sides, and it was this discovery that encouraged him to pursue mathematics rather than philology—a rather insignificant incident to be fraught with such stupendous consequences—consequences materially affecting our present progress in mental and material development.

A detailed construction of this problem by elementary geometry was first made by Pauker and Erchinger.

Gauss worked quite independently of his teachers at Göttingen, and it was while he was there as a student that he made many of his greatest discoveries in the theory of numbers, his favorite subject of investigation. Among his small circle of intimate friends was Wolfgan Bolyai, the discoverer of non-Euclidean geometry.

In 1798 Gauss returned to Brunswick, where he earned a livelihood by private tuition. Later in the year he repaired to the University of Helmstadt to consult the library, and it was while here that he made the acquaintance of Pfaff, a mathematician of great power. Laplace, when asked who was the greatest mathematician in Germany, replied, Pfaff. When the questioner said he should have thought Gauss was, Laplace replied: "Pfaff is the greatest mathematician in Germany; but Gauss is the greatest in all Europe."*

In 1799 Gauss published his demonstration that every algebraical equation with integral coefficients has a root of the form $a+bi$, a theorem of which he gave three distinct proofs. In 1801, he published *Disquisitiones Arithmeticæ*, a work which revolutionized the whole theory of numbers. "The greater part of this most important work was sent to the French Academy the preceding year, and had been rejected with a sneer which, even if the work had been as worthless as the referees believed, would have been unjustifiable."† Gauss had written far in advance of the judges of his work, and so the recognition of its merits had to wait until the mathematical world came in sight of this splendid creation. Gauss was deeply hurt because of this unfortunate incident, and it was partly due to it that he was so reluctant to publish his subsequent investigations.

The next important discovery of Gauss was in a totally different department of mathematics. The absence of a planet between Mars and Jupiter, where Bode's Law would have led observers to expect one, had long been remarked, but not until 1801 was any of the numerous groups of minor planets which occupy that space observed. On the first of January, 1801, Piazzi of Palermo discovered the first of these planets, which he called Ceres, after the tutelary goddess of Sicily.‡ While the announcement of this discovery created no great surprise, yet it was very interesting, since it occurred simultaneously

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*Cajori's *A History of Mathematics*.
†Ball's *A Short History of Mathematics*.
‡Young's *General Astronomy*, edition of 1898, page 368.

with a publication by the philosopher Hegel, in which he severely criticised astronomers for not paying more attention to philosophy, a science, said he, which would have shown them at once that there could not possibly be more than seven planets, and a study of which would have prevented, therefore, an absurd waste of time in looking for what in the nature of things could not be found. This is only one instance of the many refutations of dogmatic statements of philosophers who presage nature's laws without confirming them by actual observations.

However, the new planet was seen under conditions so unfavorable as to render it almost impossible to forecast its orbit. Fortunately the observations of the planet were communicated to Gauss. Gauss made use of the fact that six quantities known as elements completely determine the motion of a planet unaffected by perturbations. Since each observation of a planet gives two of these, e. g., the right ascension and declination, therefore three observations are sufficient to determine the six quantities and therefore to completely determine the planet's motion. Gauss applied this method and that of least squares and his his analysis proved a complete success,* the planet being rediscovered at the end of the year in nearly the position indicated by his calculations. This success proved him to be the greatest of astronomers as well as the greatest of "arithmeticians."

The attention excited by these investigations procured for him in 1807 the offer, from the Emporer of Russia, of a chair in the Academy ot St. Petersburg. But Gauss, having a marked objection to a mathematical chair, by the advice of the astronomer Olbers, who desired to secure him as director of a proposed new observatory at Göttingen, declined the offer of the emperor and accepted the position at Göttingen. He preferred this position because it afforded him an opportunity to devote all his time to science. He spent his life in Göttingen in the midst of continuous work and after his appointment never slept away from his observatory except on one occasion when he accepted an invitation from Humbolt† and attended a scientific congress at Berlin, in 1828. The only other time that he was absent from Göttingen was in 1854, when a railroad was opened between Göttingen and Hanover.‡

For some years after 1807 his time was almost wholly occupied by work connected with his observatory. In 1809 he published at Hamburg his *Theoria Motus Corporum Coelestium*, a treatise which contributed largely to the improvement of practical astronomy, and introduced the principle of curvilinier triangulation. In this treatise are found four formulæ in spherical trigonometry, commonly called "Gauss's Analogies," but which were published somewhat earlier by Karl Brandon Mollweide of Liepzig, 1774-1825, and still earlier by Jean Baptiste Joseph Delambre (1749-1822).|| On observations in general (1812-1826) we have his memoir, *Theoria Combinationis Observationum Erroribus Minimis Obnoxia*, with a second part and supplement.

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*Berry's *A Short History of Astronomy*.
†*Britannica Encyclopedia*, 9th edition, Vol. X, page 104.
‡Cajori's *A History of Mathematics*.
||Cajori's *A History of Mathematics*.

A little later he took up the subject of geodesy and from 1821 to 1848 acted as scientific adviser to the Danish and Hanoverian governments for the survey then in progress. His papers of 1843 and 1866, *Ueber Gegenstände der höhern Geodäsie*, contain his researches on the subject.

Gauss's researches on *Electricity and Magnetism* date from about the year 1830. In 1833 he published his first memoir on the theory of magnetism, the title of which is *Intensitas vis Magneticæ Terrestris ad Mensuram Absolutam Revocata.* A few months afterward he, together with Weber, invented the declination instrument and bifilar magnetometer. The same year they erected at Göttingen a magnetic observatory free from iron (as Humbolt and Arago had previously done on a smaller scale), where they made magnetic observations and showed in particular that it was possible and practical to send telegraphic signals, having sent telegraphic signals to neighboring towns. At this observatory he founded an association called the *Magnetische Varein*, composed at first almost entirely of Germans, whose continuous observations at fixed times extended from Holland to Sicily. The volumes of their publications, *Resultate aus der Beobachtungen des Magnetischen Vereins*, extend from 1833 to 1839. In these volumes for 1838 and 1839 are contained two important memoirs by Gauss, one on the general theory of earth-magnetism, the other on the theory of forces attracting according to the inverse squares of the distance. Like Poisson, he treated the phenomena in electrostatics as due to attractions and repulsions between imponderable particles. In electro-dynamics he arrived, in 1835, at a result equivalent to that given by W. E Weber in 1846, viz: that the attraction between two electrified particles, e and e', whose distance apart is r, depends on their relative motion and position according to the formula

$$ee'\, r^{-2}\{1+(rd^2r-\tfrac{1}{2}dr^2)^2c^{-2}\}.$$

Gauss, however, held that no hypothesis was satisfactory which rested on a formula and was not a consequence of physical conjecture, and as he could not form a plausible physical conjecture he abandoned the subject. Such conjectures were proposed by Riemann in 1858, and by C. Neumann and E. Betti in 1868, but Helmholtz in 1870, 1873 and 1874 showed that these conjectures were untenable.

In 1833, in a memoir on capillary attraction, he solved a problem in the Calculus of Variation, involving the variation of a certain double integral, the limits of integration also being variable; it is the earliest example of the solution of such a problem.

In 1846 was published his *Dioptrische Untersuchungen*, researches on optics, including systems of lenses.

As has already been observed, Gauss's most celebrated work in pure mathematics is the *Disquisitiones Arithmeticæ*, and a new epoch in the theory of numbers dates from the time of its publication. This treatise, Legendre's *Théorie des nombres* and Derichlet's *Vorlesunger über Zahlentheorie* are the standards on the Number Theory.

In this work Gauss has discussed the solution of binominal equations of the form $x^n=1$, which involves the celebrated theorem that the only regular polygons which can be constructed by elementary geometry are those of which the number of sides is $2^m(2^n+1)$, where m and n are integers and 2^n+1 is a prime. These equations are called *cyclotomic equations*, when n is prime and when they are satisfied by a primitive nth root of unity.

Gauss developed the theory of ternary quadratic forms involving two indeterminates, and also investigated the theory of determinants on whose results Jacobi based his researches on this subject.

The theory of Functions of Double Periodicity had its origin in the discoveries of Abel and Jacobi. Both arrived at the Theta Functions which play so large a part in the Theory of Double Periodic Functions. But Gauss had independently and at a far earlier date discovered these functions and their chief properties, having been led to them by certain integrals which occurred in the *Determinatio Attractionis*, to evaluate which he invented the transformation now associated with the name of Jacobi. In the memoir, *Determinatio Attractionis*, it is shown that the secular variations, which the elements of the orbit of a planet experience from the attraction of another planet which disturbs it, are the same as if the mass of the disturbing planet were distributed over its orbit into an elleptic ring in such a manner that equal masses of the ring would correspond to arcs of the orbit described in equal times.

Gauss's collected works have been published by the Royal Society of Göttingen, in seven 4-to volumes, 1863-1871, under the editorship of E. J. Schering. They are as follows: (1) The *Disquisitiones Arithmeticæ*, (2) *Theory of Numbers*, (3) *Analysis*, (4) *Geometry and Method of Least Squares*, (5) *Mathematical Physics*, (6) *Astronomy*, and (7) *Theoria Motus Corporium Cœlestium*. These include besides his various works and memoirs, notices by him of many of these, and of works of other authors in the *Göttingen gelehrte Anzeigen*, and a considerable amount of previously unpublished matter, *Nachlass*. Of the memoirs in pure mathematics, comprised for the most part in volumes ii, iii and iv (but to these must be added those on *Attraction* in volume v), there is not one which has not signally contributed to the branch of mathematics to which it belongs, or which would not require to be carefully analyzed in a history of the subject.

His collected works show that this wonderful mind had touched hidden laws in Mathematics, Physics and Astronomy, and every one of the subjects which he investigated was greatly extended and enriched thereby. He was also well versed in general literature and the chief languages of modern Europe, and was a member of nearly all the leading scientific societies in Europe.

He was the last of the great mathematicians whose interests were nearly universal. Since his time, the literature of most branches of mathematics has grown so rapidly that mathematicians have been forced to specialize in some particular department or departments.

Gauss was a contemporary of Lagrange and Laplace, and these three, of which he was the youngest, were the great masters of modern Analysis. In

Gauss that abundant fertility of invention which was marvelously displayed by the mathematicians of the preceding period, is combined with an absolute rigorousness in demonstration which is too often wanting in their writings. Lagrange was almost faultless both in form and matter, he was careful to explain his procedure, and, though his arguments are general, they are easy to follow. Laplace, on the other hand, explained nothing, was absolutely indifferent to style, and, if satisfied that his results were correct, was content to leave them either without a proof or even a faulty one. Many long and abstruse arguments were passed by with the remark, "it is obvious." This led Dr. Bowditch, of Harvard university, while translating Laplace's *Mechanique Céleste*, to say that whenever he came to Laplace's "it is obvious," he expected to put in about three weeks of hard work in order to see the obviousness. Gauss, in his writings, was as exact and elegant as Lagrange, but even more difficult to follow than Laplace, for he removed every trace of the analysis by which he reached his results. and even studied to give a proof which, while rigorous, should be as concise and synthetical as possible. He said: "Mathematics is the queen of the sciences, and arithmetic is the queen of mathematics." and his *Disquisitiones* confirms the statement.

Gauss had a strong will, and his character showed a curious mixture of self-conscious dignity and child-like simplicity. He was little communicative, and at times morose.

He possessed a remarkable power of attention and concentration, and in this power lies the secret of his wonderful achievements. As a proof of this power of attention we quote from Carpenter's *Mental Physiology*. Gauss, while engaged in one of his most profound investigations, was interrupted by a servant who told him that his wife (to whom he was known to be deeply attached, and who was suffering from a severe illness) was worse. "He seemed to *hear* what was said, but either did not comprehend it or immediately forgot it, and went on with his work. After some little time, the servant came again to say that his mistress was much worse, and to beg that he would come to her at once; to which he replied: 'I will come presently.' Again he lapsed into his previous train of thought, entirely forgetting the intention he had expressed, most probably without having distinctly realized to himself the import either of the communication or of his answer to it. For not long afterwards when the servant came again and assured him that his mistress was dying and that if he did not come immediately he would probably not find her alive, he lifted up his head and calmly replied, 'Tell her to wait until I come,'—a message he had doubtless often before sent when pressed by his wife's request for his presence while he was similarly engaged."

In bringing this imperfect sketch to a close, we wish to call attention to the fact that it has been conclusively shown that Gauss was not the first to give a satisfactory representation of complex numbers in a plane, this having been first satisfactorily done by Casper Wessel in 1797, though Wallis had made some use of graphic representation of complex numbers as early as 1785. Gauss

needs no undue credit to make him famous—the writing alone of any one of the seven of his collected works being sufficient to rank him among the great mathematicians of his day. However, it was Gauss who in 1831, "by means of his great reputation, made the representation of imaginery quantities in the 'Gaussian plane' the common property of all mathematicians." He brought also into general use the sign i for $\sqrt{-1}$, though it was first suggested by Euler. He called $a+bi$ a *complex number* and called a^2+b^2 the *norm*.

THE POPULARIZATION OF NON-EUCLIDEAN GEOMETRY.

By GEORGE BRUCE HALSTED.

In a charming article in the *Popular Science Monthly* for January, 1901, entitled, "Geometry: Ancient and Modern," Edwin S. Crawley delightfully helps the cultured reader to get his orientation in this subject for a start into the new century.

But strangely enough this admirable paper becomes somewhat obscure when it becomes tri-dimensional. It says: "If we proceed beyond the domain of two-dimensional geometry we merge the ideas of non-Euclidean and hyper-space."

If we do so, we are apt to blunder. Just as the Bolyai plane is utterly independent of the Euclidean plane, so the triply extended space of Bolyai is utterly independent of any Euclidean space or hyper-space.

The idea that tri-dimensional Bolyai space needs four-dimensional Euclidean space is an error into which many philosophers and some mathematicians have been led, perhaps from the unfortunate adoption of the name "radius of space curvature" for the space-constant.

This blunder was refuted even before it was born by Bolyai's geodesic geometry of limit surfaces.

Thereby a Euclidean plane can be represented by a surface in Bolyai space, the theorems of Euclidean geometry find their realization as surface theorems in non-Euclidean space, where the geodesic geometry is that of Euclidean straight lines in a Euclidean plane.

Because this error about "curvature in space" is so widespread and so insidious, I treated it fully in my "Report on Progress in Non-Euclidean Geometry" to the American Association for the Advancement of Science.

The very next sentence in Professor Crawley's article reads as follows: "The ordinary triply-extended space of our experience is purely Euclidean."

Here our author states not only something which is not known, but, strangely enough, something which never can be known, which never can be proven.

Man's metric knowledge of the world independent of man, coming through imperfect instruments, for example the eye, cannot be absolute and exact.

The pure idea of a perfect plane is a creation of the human mind.

When the variations in the approximately plane surface of an actual body are minute, we deliberately make the perceived imperfections disappear, that we may identify the surface we seem to see with our ideal creation, the perfect plane. Surface is an ideal or imaginery creation to which we fit even the apparent (not real) boundaries of physical objects.

Just so the straight line is a non-real creation.

In the theoretical, the scientific, the mathematical handling of any empirical data the process is always the same. Always the results of any observations hold good only within definite limits of exactitude and under particular conditions; we replace these results with statements of absolute precision and generality.

Our replacement is only confined in its free arbitrariness in that it should seem to snuggle to the seeming facts, and must introduce no logical contradictions.

In this sense the ordinary triply-extended space of our experience is at present Euclidean or Bolyaian or Riemannian as you choose. Each is, up to the present day, in simple and perfect harmony with experience, with experiment, with the properties of the solid bodies and the motions about us.

If the angle-sum of a single rectilineal triangle is exactly a straight angle, space is Euclidean; if less than a straight angle, Bolyaian; if more, Riemannian.

The mechanics of actual bodies in the external space of our experience might conceivably be shown by merely approximate measurements (the only kind that ever were) to be non-Euclidean; just as a body might be shown to weigh more than two grams or less than two grams, though it never can be shown to weigh precisely, absolutely two grams.

In this sense the Euclidean geometry is positively hopeless, in that it never can be proven and no respectable person would now for a moment attempt to establish it, while there was nothing theoretically absurd in the claim of C. S. Peirce to be able to show that space is Bolyaian, which claim has found its way into Boyer's *Histoire des Mathématiques*, page 247, though the index confuses C. S. Peirce with his father, Benjamin Peirce.

When Lobachevski exerted himself to obtain with exceeding great precision the sum of the three angles in the very largest triangles attainable for his measurement, he found this sum did not differ from two right angles by the hundredth part of a second. This shows that the space of experience approaches the ideal Euclidean space with an approximation which is very far-reaching. But it would be strange if any educated person should need to be told that all our measurements are approximate only, and that with the approximate we can never reach the absolutely exact.

The mistake of supposing that they *know* our space, "the space in which we really live," is not Boylyaian, made by Phillips and Fisher, Professors in

Yale, in their Elements of Geometry, page 23, and even by so good a mathematician as H. Schubert of Hamburg, I have exposed in my paper, "Non-Euclidean Geometry," AMERICAN MATHEMATICAL MONTHLY, Vol. 7, pages 123-133.

A striking testimony that the non-Euclidean geometry has won its fight and henceforth must be reckoned with is found in the History of Mathematics by Boyer, Paris, 1900. He says, pages 240-7: "The last quarter of the nineteenth century has seen built up interesting theories. But beyond contradiction the most original researches of this period pertain to the *non-Euclidean Geometries*, and it is with them that we will terminate this exposé of contemporary science."

I only wish the short account which follows this prelude and terminates the book were as accurate as it is impressive and stimulating. A full-page portrait of Lobachevski is given. But of the two entirely different likenesses we possess of the great Russian, this is the conventional one of Lobachevski depressed, baffled, about to become blind and die with his great achievement unrecognized. The other portrait, a Daguerreotype from life, which I first saw at Kazan, and of which now you may see a copy as frontispiece of Engel's magnificent "Nikolaj Iwanowitsch Lobatschefskij," is a picture of Lobachevski the fighter, the dare-devil, the irrepressible, who startled and scandalized the despotic authorities of Kazan and the University by shooting off his rocket, who contemptuously overthrew the great Legendre, of Lobachevski who knew he was right against a scornful world, who has given to us a new freedom to explain and understand our universe and ourselves.

Boyer goes on as follows: "From far in the past men have striven to demonstrate the famous axiom postulated twenty centuries ago by Euclid, to-wit: Through a point can be drawn only one parallel to a given straight. These attempts remained unfruitful. However, at the end of the eighteenth century, an Italian jesuit, Saccheri, attempted to found a geometry resting on a principle different from the celebrated postulate." This is a mistake. Saccheri's was simply one more attempt to prove the postulate, and he thought he had proven it. The title of his book is: *Euclides ab omni naevo vindicatus*. THE AMERICAN MATHEMATICAL MONTHLY has the honor of being the first to publish a translation of this now famous work into any modern language.

Boyer continues: "Finally at the beginning of the nineteenth century, a Russian, Lobachevski, and a Hungarian, John Bolyai, perceived at about the same time the impossibility of this demonstration. Their works published independently one of the other had without doubt been inspired by the doctrines of the philosopher Kant, who, in a passage of his *Kritik* of pure reason, indicated a new consideration of space. For this latter, space existed *a priori*, preceding all experience, as form completely subjective of our intuition."

The statement that the work of John Bolyai was inspired by Kant is a complete mistake. It is a gratuitous assumption cut out of whole cloth. There is nothing to show that John Bolyai ever even heard of the existence of Kant. At Maros-Vásárhely I examined the papers, the correspondence, the

Nachlass of John Bolyai. There was not the slightest mention of Kant, not the remotest reference to Kant.

Lobachevski knew of Kant through the professor of physics at Kazan, Bronner, once an admirer of the *Kritik der reinen Vernunft*. But Lobachevski tells us more than once the inspiration and mental ancestry of his achievement, and there is no place for Kant.

Kant, supposing that we knew Euclid's geometry and Aristotle's logic to be true absolutely and necessarily, accounted for the paradox by teaching that this seemingly universal synthetic knowledge was in reality particular, being part of the apparatus of the human mind itself. When Boole and De Morgan made new kinds of logic of which the Aristotelian is only a special case, when Lobachevski, Bolyai and Riemann made new kinds of geometry of which the Euclidean is only a special case, then the very foundations were cut away from under the Kantian system of philosophy, and it was left like a man trying to lift himself by his own boots.

The Scotch philosophy accounts for this supposed absolute, metrically exact knowledge, by teaching that there are in the human mind certain synthetic theorems, called by them intuitions, directly God-given. Dr. McCosh summed up this Scotch doctrine in a big book entitled, "The Intuitions of the Mind, Inductively Investigated."

One of these divinely implanted intuitions was Euclid's famous parallel-postulate!! *Voila!*

"Yet," to quote a sentence from Wenley's criticism in Science of McCosh's disciple Ormond, "we may well doubt whether a thinker standing with one foot firmly planted on the Rock of Ages, and the other pointing heavenward, has struck the attitude most conducive to progress."

After a brief biography of Lobachevski and a quotation from his Introduction to New Principles of Geometry, which Introduction I have translated into English and published as volume V of the Neomonic Series, Boyer continues: "This postulated, behold how he proceeds to the development of his doctrine. He announces at the beginning the following axiom: Through a point can be drawn *many parallels* to a given straight." To say the least this is likely to produce misconception.

Lobachevski assumes that through a given point outside a given straight can be drawn many straights which never meet the given straight, but of this sheaf of non-meeters only two are parallel to the given straight, namely, the two which approach the given straight asymptotically.

Again Boyer says, page 245: "He became professor, and when he died in 1856 he occupied the position of Rector of the University where he had entered as simple student." This is another mistake. When Lobachevski died in 1856 he had been displaced from his position as Rector for ten years.

Passing on to the Riemannian geometry, Boyer says: "To construct this Geometry, its inventor rejects the postulatum and the first axiom of Euclid: two points determine a straight." In fact neither of these assumptions of Euclid

need be rejected to get a Riemannian geometry, the "Single Elliptic Geometry."

Boyer has perhaps been misled by his own paraphrases for what really occurs in Euclid. The real postulatum is as follows: "And if a straight cutting two straights makes with them angles interior and lieing on the same side, which together are less than two right angles, then the two straights indefinitely produced cut each other on the side on which these angles lie."

Needless to say, this remains true in both single and double elliptic geometry. The postulate currently taken in place of the unwieldy postulatum of Euclid is "that two straight lines, which cut one another, cannot be both parallel to the same straight line," which is credited by Playfair to Ludlam, though attributed even by Cajori to Playfair, and currently called Playfair's axiom. Even this does not help toward Riemannian spaces, for in them there are no parallels in the sense of coplanar non-meeting straights.

Finally Boyer gives us the well-known confusion in connection with Beltrami's pseudo-sphere.

John Bolyai found a surface in Lobachevskian space whose geodesic geometry is that of Euclid's straights. No one supposed that this reduced Euclidean to be only a branch of Bolyaian geometry.

Beltrami found a surface in Euclidean space whose geodesic geometry is that of Bolyai's straights. Boyer says, page 246, this reduced the Geometry of Lobachevski to be only a branch of ordinary geometry. On the contrary, the truth is that Euclidean geometry is only that special case of Bolyaian geometry made by assuming the space-constant as infinite.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

134. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A certain piece of land is surrounded by a four-board fence, the boards being 16 feet long. The number of acres in the land equals the number of boards in the fence. How many acres in the land?

I. Solution by **G. B. M. ZERR**, A. M., Ph. D., The Temple College, Philadelphia, Pa.; **P. S. BERG**, B. Sc., Larimore, N. D.; and **MARTIN SPINKS**, Wilmington, O.

(1) *When the field is in the form of a square.*

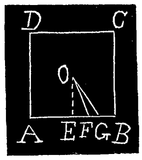
Let $ABCD$ be the square, O its center, $FG=l=16$ feet=the length of a panel of the fence, and $n=4$=the number of boards in a panel. Then the area of the triangle $OFG=\frac{1}{2}OE\times FG=\frac{1}{2}OE.l=n$ acres.

$\therefore\ OE=\dfrac{2\times43560\times n}{l}$, and AB, a side of the field$=2OE=\dfrac{4\times43560\times n}{l}$.

Hence the area of the field$=\left(\dfrac{4\times43560\times n}{l}\right)^2\div43560=\dfrac{16\times43560\times n^2}{l^2}$.

$=43560$ acres.

SPINKS.



$(\text{side})^2/160=$number of acres in tract.

$4\times16\frac{1}{2}\times\text{side}=$number of feet in perimeter of field.

$4\left(\dfrac{4\times16\frac{1}{2}\times\text{side}}{16}\right)=$number of boards in fence.

$\therefore\ \dfrac{(\text{side})^2}{160}=\dfrac{4(4\times16\frac{1}{2}\times\text{side})}{16}=16\frac{1}{2}\times\text{side}.$

$\therefore$ side$=2640$ rods$=8\frac{1}{4}$ miles.

$(2640)^2\div160=43560$ acres.

ZERR AND BERG.

(2) *When the field is in the form of a circle.*

Let $R=$radius in rods.

$\therefore\ \pi R^2/160=$area in acres.

$2\pi R\times16\frac{1}{2}=$perimeter in feet.

$4\left(\dfrac{2\pi R\times16\frac{1}{2}}{16}\right)=$number of boards in perimeter.

$\therefore\ \dfrac{\pi R^2}{160}=\dfrac{4(2\pi R\times16\frac{1}{2})}{16}=8\frac{1}{4}\pi R.$

$\therefore\ R=1320$ rods.

$\pi R^2/160=10890\pi$ acres.

II. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Let $16m(p^2+q^2)x$ and $16nx=$the respective sides, in feet, of a *parallelogram.* Then the number of boards in the fence

$$=4\times\tfrac{1}{16}\times32[m(p^2+q^2)+n]x=8[m(p^2+q^2)+n]x\ldots.(1).$$

For *rectangles*, the number of acres

$$=\frac{(16nx)[16m(p^2+q^2)x]}{43560}=\frac{32mn\,(p^2+q^2)x^2}{5445},$$

which equals (1).

$$\therefore\ x=\frac{5445[m(p^2+q^2)+n]}{4mn(p^2+q^2)}\ldots.(2).$$

For *rhomboids*, the altitude$=16m(p^2-q^2)x$ or $32mpqx$; and number of acres

$$=\frac{32mn\,(p^2-q^2)x^2}{5445}\ \text{or}\ \frac{64mnpqx^2}{5445}.$$

Equating these two values with (1) and reducing, we find

$$x=\frac{5445[m(p^2+q^2)+n]}{4mn(p^2-q^2)}\ldots.(3), \text{ and } x=\frac{5445[m(p^2+q^2)+n]}{8mnpq}\ldots.(4).$$

p, q, and m may be any integers. $p>q$. Also, p and q may be such fractions that, in connection with m and n, will render x integral or of such values as to have the sides divisible by 16.

(a) Take $p=\frac{4}{5}$, $q=\frac{3}{5}$, $m=1$; and substitute in (2), (3), and (4).

Then $x=\dfrac{5445(1+n)}{4n}\ldots.(5)$, $x=\dfrac{136125(1+n)}{28n}\ldots.(6)$,

$x=\dfrac{45375(1+n)}{16n}\ldots.(7)$, and (1) becomes $8(1+n)x$.

x must be integral.

From the factors of 5445, we observe that, in (5), $n=3$, 11, 15, 55, 99, 363, $\frac{1}{3}$, $\frac{5}{3}$, $\frac{3}{5}$, $\frac{11}{5}$, $\frac{5}{11}$, $\frac{9}{11}$, etc.; in (6), $n=55$, 363, $\frac{1}{55}$. etc.; and in (7), $n=15$, $\frac{1}{15}$, etc.

Tne following table of values explains itself.

n	*Eq.*	x	*Boards=Acres*	*Sides in Feet.*
3	(5)	1815	58080	29040 and 87120
11	(5)	1485	142560	23760 and 261360
$\frac{9}{11}$	(5)	3025	44000	48400 and 39600
15	(5)	1452	185856	23232 and 348480
15	(7)	3025	387200	48400 and 726000
55	(5)	1357	620928	22176 and 1219680
55	(6)	4950	2217600	79200 and 4356000
etc.	etc.	etc.	etc.	etc.

(b) Take $p=2$, $q=m=1$; and substitute in (2), (3), and (4).

Then $x=\dfrac{1089(5+n)}{4n}\ldots.(8)$, $x=\dfrac{1815(5+n)}{4n}\ldots.(9)$, $x=\dfrac{5445(5+n)}{16n}\ldots(10)$, and (1) becomes $8(5+n)x$.

Whence the following table:

n	*Eq.*	x	*Boards=Acres*	*Sides in Feet*
3	(8)	726	46464	58080 and 34848
3	(9)	1210	77440	96800 and 58080
11	(8)	396	50688	31680 and 69696
11	(9)	660	84480	52800 and 116160
11	(10)	605	77440	48400 and 106480
$\frac{9}{11}$	(8)	1331	61952	106480 and 17424
$\frac{9}{11}$	(10)	2420	112640	193600 and 31680
etc.	etc.	etc.	etc.	etc.

(*c*) And so on for all other values of p, q, and m.

Mr. Gruber sent in three different solutions, and Mr. D. B. Northrup, of Mandana, N. Y., sent in the results for tracts in the form of (1) circle, (2) square, (3) rectangle sides as 2:1, (4) a triangle ratios of side 6:6:7, and (5) an ellipse having a major-axis double the minor-axis. Other very simple solutions of the problem are possible when the tract is in the shape of a square, but the solutions above are quite sufficient for all purposes. ED. F.

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

What is the size of the smallest cubical box, inside dimension, that will contain four balls each ten inches in diameter?

Solution by H. N. DAVIS, (Brown University), 159 Brown Street, Providence. R. I.

Let the base of the box be of such a size that two of the balls when placed with their centers along a diagonal will be tangent to each other and to the sides of the box. Let their centers be C and C', the diagonal AD, and the center O.

Draw BC perpendicular to AE. Then $AB=BC=r=5$.

$\therefore AC=5\sqrt{2}$.

$AO=5\sqrt{2}+CO=5\sqrt{2}+r=5\sqrt{2}+5$.

$AD=10(1+\sqrt{2})=\sqrt{2}AE$.

$AE=5\sqrt{2}+10=17.07106$ inches (nearly).

If the second layer of two balls be placed along the other diagonal (at $M+M'$) the position of M with reference to C and the side AF considered as a base will be exactly the same as that of C and C' with reference to AD. The figure may then be taken as an elevation and the height of the box will be exactly equal to its base-edge, and the box will be a cube. Q. E. D.

Good solutions were received from *PROF. J. M. STRASBURG*, Chicago, Ill., *D. B. NORTHRUP*, *G. B. M. ZERR*, and *M. A. GRUBER*.

ALGEBRA.

112. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In *Hall and Knight's Higher Algebra* I find the following:

If $a+b+c=0$, then

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2};\qquad \frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2};$$

and if $a+b+c+d=0$, then

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

QUERY. Is there a general law governing such expressions? Investigate.

I. Solution by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

Consider the following general equation with its second term missing, thus

$$f(x)=x^n+m_2x^{n-2}-m_3x^{n-3}\dots m_{n-1}x+(-1)^nm_n=0\dots(1).$$

Suppose the roots of this equation to be $a_1,\ a_2,\ a_3,\ \dots a_n$.

Then $f'(x)=\dfrac{f(x)}{x-a_1}+\dfrac{f(x)}{x-a_2}+\dfrac{f(x)}{x-a^3}\dots\dfrac{f(x)}{x-a_n}\dots(2)$.

[*Hall and Knight's Higher Algebra*, page 468.]

Divide (1) by $x-a_1$,

$$\frac{f(x)}{x-a_1}=x^{n-1}+a_1x^{n-2}+(m_2+a_1^2)x^{n-3}-(m_3-a_1m_2-a_1^3)x^{n-4}\dots;$$

the general term being

$$(-1)^r[m_r-a_1m_{r-1}\dots(-1)^ra_1^{r-2}m_2+(-1)^ra_1^r]x^{n-(r+1)};$$

and similar expressions for

$$\frac{f(x)}{x-a_2},\ \frac{f(x)}{x-a_3}\dots\frac{f(x)}{x-a_n}.$$

Finding the value of the right-hand member of (2) by addition of the expressions just obtained, and adopting the notation

$$_h=\frac{a_1^h+a_2^h\dots a_n^h}{h},$$

we get, $f'(x)=nx^{n-1}+m_2(n-2)x^{n-3}-m_3(n-3)x^{n-4}\dots m_{n-2}x+(-1)^{n-1}m_{n-1}$

$$=nx^{n-1}+(nm_2+2S_2)x^{n-3}-(nm_3-3S_3)x^{n-4}+(nm_4+2m_2S_2+4S_4)x^{n-5}\dots;$$

the general term being

$$(-1)^r[nm_r+2m_{r-2}S_2-3m_{r-3}S_3\dots(-1)^r(r-2)m_2S_{r-2}+(-1)^rrS_r].$$

Whence, by equating coefficients, and solving for $m_2,\ m_3\ \dots m_{n-1}$) we obtain,

$$m_2=S_2\dots(5);\ m_3=S_3\dots(6);\ m_4=\frac{-(2m_2S_2+4S_4)}{4}\dots(7);$$

$$m_5=\frac{-(2m_3S_2-3m_2S_3-5S_5)}{5}\dots(8);$$

and in general,

$$m_r=\frac{-[2m_{r-2}S_2-3m_{r-3}S_3\dots(-1)^r(r-2)m_2S_{r-2}+(-1)^rrS_r]}{r}\dots(9).$$

Multiplying (1) by x^{k-n}, we have

$$x^k+m_2x^{k-2}-m_3x^{k-3}\dots(-1)^r m_r x^{k-r}\dots(-1)^n m_n x^{k-n}=0.$$

Substituting a_1, a_2, $a_3\dots a_n$ each separately for x, and adding the identities thus obtained, we have

$$kS_k+(k-2)m_2S_{k-2}-(k-3)m_3S_{k-3}\dots(-1)^r(k-r)m_rS_{k-r}\dots(-1)^n\dots(10).$$

$(k-n)m_nS_{k-n}=0$.

By substituting the value of m_2 in (7) we obtain $S_2^2/2-S_4=m_4$.

In like manner, by successive substitutions, m_5, $m_6\dots m_{n-1}$ can be obtained in terms of S_2, $S_3\dots S_{n-1}$.

Hence by substitution in (10) we may obtain a relation satisfying the conditions of the problem, for any positive integral value of n.

When $n=3$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3};$$

putting $k=5$, then $S_5=S_3\times S^2$; when $k=7$, then $S_7=S_5\times S_2$.

. . . . . .

When $n=4$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3}+(k-4)(S_4-S_2^2/2)S_{k-4};$$

putting $k=5$, $S_5=S_3\times S_2$.

. . . . . .

When $n=5$, we have

$$kS_k-(k-2)S_2S_{k-2}-(k-3)S_3S_{k-3}+(k-4)$$
$$(S_2^2/2-S_4)S_{k-4}-(S_5-S_2S_3)(k-5)S_{k-5}=0.$$

II. Solution by **ROBERT B. HAYWARD**, Ashcombe, Shanklin, Isle of Wight.

S_r denotes $a^r+b^r+c^r$.

Then $S_1=a+b+c=0$.

$S_2=a^2+b^2+c^2=-2(bc+ca+ab)$.

$S_3=a^3+b^3+c^3=3abc$.

Hence a, b, c are the roots of the equation.

$$x^3-\frac{S_2}{2}.x-\frac{S_3}{3}=0.$$

$$\therefore\ x^n-\frac{S_2}{2}.x^{n-2}-\frac{S_3}{3}x^{n-3}=0.$$

Substitute a, b, c successively for x, and add, then

$$S_n-\frac{S_2}{2}.S_{n-2}-\frac{S_3}{3}.S_{n-3}=0.$$

Whence S_n is determinable in powers of $\frac{S_2}{2}$ and $\frac{S_3}{3}$.

Hence $S_5=\frac{S_2}{2}.S_3+\frac{S_3}{3}.S_2=\frac{5}{6}S_3.S_2$, or $\frac{S_5}{5}=\frac{S_3}{3}.\frac{S_2}{2}$.

$$S_7=\frac{S_2}{2}.S_5+\frac{S_3}{3}.S_4=\frac{S_2}{2}.S_5+\frac{S_3}{3}\left(\frac{S_2}{7}\right)^2=5\left(\frac{S_2}{2}\right)^2.\frac{S_3}{3}+2.\frac{S_3}{3}\left(\frac{S_2}{2}\right)$$

$$\text{or } \frac{S_7}{7}=\frac{S_5}{5}.\frac{S_2}{2}.$$

It also follows that

$$\frac{S_9}{9}=\frac{S_2}{2}.\frac{S_7}{7}+\tfrac{1}{3}\left(\frac{5_3}{3}\right)^3\text{or }\left(\frac{S_2}{2}\right)^3\frac{S_3}{3}+\tfrac{1}{3}\left(\frac{S_3}{3}\right)^3.$$

The form of the expression for $\frac{S_n}{n}$ (putting α_2, α_3 for $\frac{S_2}{2}.\frac{S_3}{3}$) is

$$A\alpha_3{}^p\alpha_2{}^q+B\alpha_3{}^{p-2}\alpha_2{}^{q+3}+C\alpha_3{}^{p-4}\alpha_2{}^{q+6}+\ldots.$$

continued as far as the indices remain positive.

If n is of the form $6m$, $p=2m$, $q=0$.

$6m-1$, $p=2m-1$, $q=1$.
$6m+1$, $p=2m-1$, $q=2$.
$6m-2$, $p=2m-2$, $q=2$.
$6m+2$, $p=2m$, $q=1$.
$6m-3$, $p=2m-1$, $q=0$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1). Put $a+b+c=0=p$, $ab+ac+bc=q$, $abc=r$.

Then $(1+ax)(1+bx)(1+cx)=1+px+qx^2+rx^3=1+qx^2+rx^3$.

Taking logarithms and equating the coefficients of x^n, we have $\frac{(-1)^{n-1}}{n}(a^n+b^n+c^n)$ for the coefficient equal to the coefficient of x^n in

$$(qx^2+rx^3)-\tfrac{1}{2}(qx^2+rx^3)+\ldots\pm(1/n)(qx^2+rx^3)^n\ldots.$$

Let $n=2m+1$ and 2, respectively, and we get,

$$\frac{1}{2m+1}(a^{2m+1}+b^{2m+1}+c^{2m+1})=\pm q^{m-1}r,$$

$$\frac{1}{2m-1}(a^{2m-1}+b^{2m-1}+c^{2m-1})=\mp q^{m-2}r,$$

$$\tfrac{1}{2}(a^2+b^2+c^2)=-q.$$

$$\therefore \frac{a^{2m+1}+b^{2m+1}+c^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}}{2m-1}.\frac{a^2+b^2+c^2}{2}.$$

When $m=2$, 3, we get the results in the problem.

(2). Similarly, $(1+ax)(1+bx)(1+cx)(1+dx)=1+qx^2+rx^3+sx^4$.

$\therefore \dfrac{(-1)^{n-1}}{n}(a^n+b^n+c^n+d^n)$ is equal to the coefficient of x^n in

$$(qx^2+rx^3+sx^4)-\tfrac{1}{2}(qx^2+rx^3+sx^4)^2+\ldots\ldots\pm(1/n)(qx^2+rx^3+sx^4)^n.$$

$\therefore$ As before

$$\frac{a^{2m+1}+b^{2m+1}+c^{2m+1}+d^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}+d^{2m-1}}{2m-1}.\frac{a^2+b^2+c^2+d^2}{2}.$$

The same reasoning will lead to the following :

$$\frac{a^{2m+1}+b^{2m+1}+\ldots\ldots+k^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+\ldots\ldots+k^{2m-1}}{2m-1}.\frac{a^2+b^2+\ldots\ldots+k^2}{2}.$$

Also solved by *J. M. BOORMAN*, *J. SCHEFFER*, and the *PROPOSER*.

GEOMETRY.

139. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If $x^2+y^2=1$ [x and y being points corresponding to complex numbers], prove that x and y are at the ends of conjugate radii of an ellipse whose foci are ± 1. [From *Harkness and Morley's Introduction to the Theory of Functions.*]

Solution by J. W. YOUNG, Oliver Fellow in Mathematics, Cornell University, Ithica, N. Y., and FRANK GIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

Let $x=h+ik$, $y=m+in$.

The condition $x^2+y^2=1$, gives on equating real and imaginary parts,

$$h^2+m^2-k^2-n^2=1\ldots.(1),\ hk+mn=0\ldots.(2).$$

Now if the points (h, k) and (m, n) are the extremities of conjugate radii of an ellipse, we may write

$$\left.\begin{matrix} h=a\cos\phi \\ k=b\sin\phi \end{matrix}\ \begin{cases} m=a\sin\phi \\ n=-b\cos\phi \end{cases}\right\}\ (a,\ b \text{ semi-axes of ellipse}).$$

These values satisfy condition (2).

Substituting in (1) we have

$$a^2-b^2=1\ldots.(3).$$

If e is the eccentricity of the ellipse $b^2=a^2(1-e^2)$, substituting in (3) and reducing, we have $ae=\pm1$, *i. e.* the foci of the ellipse are at the points ±1.

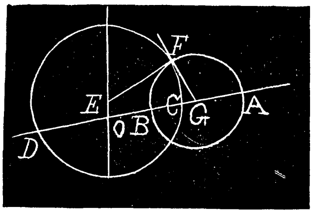
Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

140. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Having given two points on a range and a point that bisects the distance between two other points that form an harmonic ratio with the given points, give, if possible, a geometrical construction for locating the other two points.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the two given points on the range, O the point bisecting the distance between the other two points. Through O draw OE perpendicular to ABO, and on AB as diameter describe a circle AFB. Draw EF tangent to AFB, and with E as center and a radius equal to EF describe a circle cutting AO in C and D. (F is the point of tangency of EF, and EF must be greater than EO). Then C, D are the two points required.



For $GF^2=AG^2=GC.GD$.

$\therefore\ AG:GC=GD:AG$.

But $AG+GC:AG-GC=GD+AG:GD-AG$.

$\therefore\ AC:CB=AD:BD$. Q. E. D.

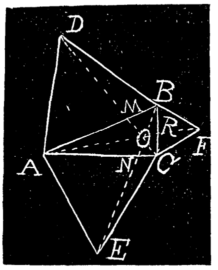
141. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

The equilateral triangle described on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles described on the other two sides.

Prove without the aid of the famous Pythagorean proposition.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and NELSON L. RORAY, Bridgeton, N. J.

Let ABC represent the right triangle, and D, E, F the vertices of the equilateral triangles constructed on the three sides.



It is seen at once that $\triangle ACF=\triangle BCE=\frac{1}{2}\triangle ABC$.

$\therefore\ \triangle ACF+\triangle BCE=\triangle ABC$.

$\triangle BDC=\triangle ABF$, $\triangle ADC=\triangle AEB$.

$\therefore\ \triangle ABD+\triangle ABC=\triangle ABF+\triangle AEB$.

$\triangle ABC=\triangle ACF+\triangle BCE$.

$\therefore\ \triangle ABD+2\triangle ABC=(\triangle ABF+\triangle ACF)+(\triangle AEB+\triangle BCE)$.

$\therefore\ \triangle ABD+2\triangle ABC=\triangle BCF+\triangle ABC+\triangle ACE+\triangle ABC$.

$\therefore \triangle ABD = \triangle BCF + \triangle ACE$. Q. E. D.

That the three lines AF, BE, and CD intersect in one point may be proved as follows:

Since EOB cuts the sides of $\triangle ACM$ in the points N, O, B, $CN\times OM \times AB = AN\times CO\times BM$....(1).

Since AOF cuts the sides of $\triangle BCM$ in O, P, F, $CO\times BP\times AM = OM\times CP \times AP$....(2).

Multiplying (1) by (2) and canceling, $CN\times BP\times AM = AN\times BM\times CP$.

Also demonstrated by *G. B. M. ZERR*, and the *PROPOSER*.

142. Proposed by WILLIAM HOOVER. A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that an infinite number of triangles can be inscribed in $x^2/a^2+y^2/b^2-1=0$ whose sides touch $a^2x^2+b^2y^2=\dfrac{a^4b^4}{(a^2+b^2)^2}$.

I. Solution by the PROPOSER.

The curves are $b^2x^2+a^2y^2-a^2b^2=0$....(1),

and $a^2(a^2+b^2)^2x^2+b^2(a^2+b^2)^2y^2-a^4b^4=0$....(2).

The invariants of (1) are $\triangle'=-a^4b^4$,

$$\theta'=-a^2b^2(a^4+a^2b^2+b^4)^2$$

and of (2), $\triangle=-a^6b^6(a^2+b^2)^4$,

$$\theta'=-2a^4b^4(a^2+b^2)^2(a^4+a^2b^2+b^4).$$

By the usual theory, the conditions of the problem are fulfilled if $\theta^2=4\triangle\theta'$....(3), which is easily seen to be the case here.

II. Solution by GEORGE A. OSBORNE, Professor of Mathematics, Massachusetts Institute of Technology, Boston, Mass.

The problem is a special case of the following:

An infinite number of triangles can be inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ whose sides touch $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}=1$, provided $\dfrac{a'}{a}+\dfrac{b'}{b}=1$.

The problem is considered in Salmon's *Conic Sections*, Art. 376, page 342, for the general form of the conic.

$$S=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$
$$S'=a'x^2+b'y^2+c'z^2+2f'yz+5g'zx+2h'xy=0.$$

The condition that an infinite number of triangles may be inscribed in S and circumscribed about S', is $\Theta'^2=4\Delta'\Theta$....(1).

Δ, Δ', are the discriminants of S, S'.

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \Delta'=\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

$$\Theta=Aa'+Bb'+Cc'+2Ff'+2Gg'+2Hh',$$

$$\Theta'=A'a+B'b+C'c+2F'f+2G'g+2H'h,$$

where A, B, C, etc., are minors of Δ, and A', B', C', etc., are minors of Δ'.

For $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ and $\dfrac{x^2}{a'^2}+\dfrac{y^2}{b'^2}-1=0$, we find $\Delta'=-\dfrac{1}{a'^2b'^2}$,

$$\Theta=-\frac{1}{a^2\,b'^2}-\frac{1}{a'^2\,b^2}-\frac{1}{a^2\,b^2}, \quad \Theta'=-\frac{1}{a'^2\,b^2}-\frac{1}{a^2\,b'^2}-\frac{1}{a'^2b'^2}.$$

Substituting in (1), we have

$$\left(\frac{a'^2}{a^2}+\frac{b'^2}{b^2}+1\right)^2=4\left(\frac{a'^2}{a^2}+\frac{b'^2}{b^2}+\frac{a'^2b'^2}{a^2\,b^2}\right)$$

from which $\dfrac{a'}{a}\pm\dfrac{b'}{b}=\pm1$.

The only real solution for ellipses is

$$\frac{a'}{a}+\frac{b'}{b}=1.$$

In problem 142,

$$a'=\frac{ab^2}{a^2+b^2},\ b'=\frac{a^2b}{a^2+b^2},\text{ from which }\frac{a'}{a}+\frac{b'}{b}=1.$$

A very excellent demonstration was received from *G. B. M. ZERR.*

CALCULUS.

NOTE ON CENTER OF CURVATURE.

By GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The fact, that the point of intersection of two normals which approach each other is neither at infinity nor at the foot of the normals when they become coincident, is not plain to most students. The difficulty may be overcome, in some instances, in the following manner.

Let (x_1, y_1), (x_2, y_2) be two points of the curve ; m_1; m_2 the slope of the tangents at these points. The equations of the normals are then

$$x+m_1y=x_1+m_1y_1,\ x+m_2y=x_2+m_2y_2.$$

It is proposed to find the coördinates of the intersection of these lines when (x_1, y_1) approaches and becomes coincident with (x_2, y_2). Eliminating x between the equations, we have

$$y=y_1+\frac{1+m_2\left(\dfrac{y_1-y_2}{x_1-x_2}\right)}{\dfrac{m_1-m_2}{x_1-x_2}}.$$

Since $\underset{x_1=x_2}{\text{Lim.}}\left(\dfrac{y_1-y_2}{x_1-x_2}\right)=\dfrac{dy_1}{dx_1}$ and $\underset{x_1=x_2}{\text{Lim.}}\left(\dfrac{m_1-m_2}{x_1-x_2}\right)=\dfrac{d^2y_1}{dx_1^2}$, we have

$$y=y_1+\frac{1+\left(\dfrac{dy_1}{dx_1}\right)^2}{\dfrac{d^2y}{dx^2}},\ \text{and}\ x=x_1-\frac{dy_1}{dx_1}\left\{\frac{1+\left(\dfrac{dy_1}{dx_1}\right)^2}{\dfrac{d^2y_1}{dx_1^2}}\right\}.$$

105. Proposed by **CHARLES C. CROSS**, Meridithville, Va.

From all points in a straight line passing through the center of a given circle tangents are drawn to the circle. If the bases and vertices of all the angles thus formed are made to coincide; required the equation of the curve passing through the tangent points.

Solution by **J. W. YOUNG**, Oliver Graduate Scholar in Mathematics, Cornell University, Ithica, N. Y.

Let the circle be $x^2+y^2=a^2$, and the given line $y=0$. Then the length of the tangent from any point $(x_1, 0)$ on the given line is $\sqrt{(x_1^2-a^2)}$.

Also the slope of the tangent may be calculated from the equation $mx_1\pm a\sqrt{(1+m^2)}=0$, which is obtained by substituting the point $(x_1, 0)$ in the tangent equation $y=mx\pm a\sqrt{(1+m^2)}$. Solving for m, we have

$$m^2=\frac{a^2}{x_1^2-a^2},\ \text{or}\ m=\pm\frac{a}{\sqrt{(x_1^2-a^2)}}.$$

To determine the required locus, use polar coördinates, with the common vertex of angles as pole and their common base as initial line; the coördinates (r, θ) of the tangent points are then given by the equations

$$\left.\begin{aligned} r&=\sqrt{(x_1^2-a^2)} \\ \tan\theta&=\frac{\pm a}{\sqrt{(x_1^2-a^2)}} \end{aligned}\right\}$$

Hence, the required locus is $r\tan\theta=\pm a$; in Cartesian coördinates this becomes $x^2(y^2-a^2)+y^4=0$.

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

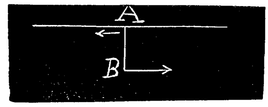
MECHANICS.

102. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A heavy particle with a light string attached is placed on the edge of a smooth table. A boy, holding the string horizontally, runs at right angles to the string. Determine the motion of the particle (1) when the boy runs with a uniform velocity; (2) when he runs with a uniform acceleration.

Solution by the PROPOSER.

This problem is one of a class in which a clear idea of the motion may be gained without solving analytically.



Let A be the position of the particle at the start, B the position of the boy's hand. We shall suppose the boy runs to the right. Now the effect upon the motion of the particle will be the same if, instead of supposing the boy to run to the right, we impose upon space a hypothetical motion to the left, in such a way as to bring the boy's hand at B at rest. This of course necessitates imposing upon the particle at A the same motion that is imposed upon space.

In our first case thus we impose upon A a uniform velocity to the left. We have thus a particle acted upon by gravity, and moving about a fixed point with uniform velocity, *i. e.* a simple conical pendulum. So the result to (1) is that as the boy runs with uniform velocity the particle will move around his hand as a simple conical pendulum.

In our second case impose upon A a constant acceleration to the left. Our particle is thus acted on by two forces, constant in direction and magnitude, viz. gravity and this hypothetical horizontal force. It will therefore oscillate horizontally and vertically about B. Also since these two forces may be combined into a single oblique force constant in magnitude and direction, these two oscillations will be equivalent to a single simple oscillation obliquely about B. The result to (2) then, is that as the boy runs with uniform acceleration the particle instead of moving around his hand as a conical pendulum, will oscillate obliquely behind it.

103. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Given the lengths a, b of the sides of a parallelogram, the direction of side a, and the position of the centroid. Prove that the locus of the foci of the ellipse of gyration at the centroid is a Cassinian Oval, having its foci distant $a/2\sqrt{3}$ from the centroid, and the constant product of its focal distances equal to $\frac{1}{12}b^2$.

Solution by the PROPOSER.

Let A, B be the principal moments of inertia, m the mass of the parallelogram, then $x^2/A+y^2/B=1/m$ is the equation to the ellipse of gyration at the centroid.

Eccentricity $=\sqrt{\dfrac{A-B}{A}}$, semi-major axis$=\sqrt{\dfrac{A}{m}}$.

$\therefore$ Distance of focus from center $=\sqrt{\dfrac{A-B}{m}}$.

Let θ=the angle the principal axes make with the sides. then if u, v be the coördinates of the focus, we easily get

$$u=\sqrt{\frac{A-B}{m}}\cos\theta,\quad v=\sqrt{\frac{A-B}{m}}\sin\theta.$$

$\therefore\ u^2+v^2=\dfrac{A-B}{m},\ u^2-v^2=\dfrac{A-B}{m}\cos2\theta.$

From problem 94, solution on page 48, Vol. VII, No. 2, we get

$$u^2+v^2=\tfrac{1}{12}\sqrt{[a^4+b^4+2a^2b^2\cos2\beta]}.$$
$$u^2-v^2=\tfrac{1}{12}(a^2+b^2\cos2\beta).$$

Eliminating $\cos2\beta$ we get

$$144(u^2+v^2)^2=24a^2(u^2-v^2)-a^4+b^4.$$

Let $u=r\cos\varphi$, $v=r\sin\varphi$.

$\therefore\ 144r^4=24a^2r^2\cos2\varphi-a^4+b^4$, or $r^4=\frac{1}{6}a^2r^2\cos2\varphi-a^4/144+b^4/144$.

If $a=b$, $r^2=\frac{1}{6}a^2\cos2\varphi$.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridge is $p_1=80$ tons. The track is supposed to be straight and practically horizontal. Had the locomotive and tender been running at the rate of $r=60$ miles an hour, how many tons would the pressure on the bridge have been?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$p_1=80$ tons$=W=mg$.

Both m and g are constant.

$\therefore$ The pressure is the same, 80 tons, no matter what the velocity.

DIOPHANTINE ANALYSIS.

83. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find three numbers in arithmetical progression whose sum is a square and cube.

I. Solution by J. W. YOUNG, Cornell University, Ithica, N. Y.; B. L. REMICK, Bradley Polytechnic Institute, Peoria, Ill.; and ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.

A number which is a square and a cube is a sixth-power.

Also three numbers in arithmetical progression may be represented by

$$a-d,\ a,\ a+d;\ \text{whose sum is } 3a.$$

These considerations lead at once to the following expressions for the required numbers : 3^5x^6-d, 3^5x^6, 3^5x^6+d, where x, d, are any numbers.

The sum is evidently $(3x)^6=[(3x)^2]^3=[(3x)^3]^2$.

As an example we may take $x=1$, $d=100$.

$$143,\ 243,\ 343, \text{ whose sum}=729=27^2=9^3.$$

II. **Solution by the PROPOSER.**

Let $\frac{1}{3}(x-y)^2$, $\frac{1}{3}(x^2+y^2)$, $\frac{1}{3}(x+y)^2$ be the three numbers.

Their sum is x^2+y^2.

Let $x^2+y^2=a^6m^6$. Let $x=m^2-n^2$, $y=2mn$.

$\therefore\ x^2+y^2=(m^2+n^2)^2=a^6m^6$.

$\therefore\ m^2+n^2=a^3m^3$.

Let $n=pm$.

$\therefore\ m^2(1+p^2)=a^3m^3$.

$\therefore\ m=\dfrac{1+p^2}{a^3},\ n=\dfrac{p(1+p^2)}{a^3}$.

$$x=\frac{(1+p^2)^2(1-p^2)}{a^6},\quad y=\frac{2p(1+p^2)^2}{a^6},\quad x^2+y^2=\frac{(1+p^2)^6}{a^{12}}.$$

$\therefore$ The numbers are

$$\tfrac{1}{3}\left(\frac{(1+p^2)^2(1-2p-p^2)}{a^6}\right)^2,\quad \tfrac{1}{3}\left(\frac{(1+p^2)^6}{a^{12}}\right),\quad \tfrac{1}{3}\left(\frac{(1+p^2)^2(1+2p-p^2)}{a^6}\right)^2.$$

Also solved by *CHAS. C. CROSS*, *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, *J. SCHEFFER*, and *ELMER SCHUYLER*.

84. **Proposed by the late SYLVESTER ROBINS, North Branch Depot, N. J.**

The nth term of an infinite series of "nests" contains all the prime, integral, rational parallelopipeds that have 3^n for their solid diagonals. It is required to determine the general expression for N=the number of such solids in nth term, and to exhibit the dimensions of all the "eggs" in the first six nests.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In parallelopipeds the square of the solid diagonal equals the sum of the squares of the three dimensions (length, breadth, and height). Whence, $(3^n)^2=$ the sum of three squares.

When $(3^n)^2$=the sum of three *integral* squares, I have found, by inspection, that the entire number of sets of three sqnares is, in terms of n, $\dfrac{3^n+2n-1}{4}$, of which $\dfrac{3^{n-1}+1}{2}$ are prime sets, and $\dfrac{3^{n-1}+2n-3}{4}$ multiple sets.

$\therefore$ According to the problem, $n=\dfrac{3^{n-1}+1}{2}$.

In *prime* sets of the sum of three squares equal to a square, it will be observed that two of the three squares are even and the other odd.

By means of an extensive table containing the odd numbers that are equal

to the sum of two squares, I have found the following complete sets of "eggs" for each nest.

In nest 1, where $n=1$ and $N=1$: 1, 2, 2. PROOF.—$1^2+2^2+2^2=(3^1)^2=9$.

In nest 2, where $n=2$ and $N=2$: 1, 4, 8; 7, 4, 4.

In nest 3, where $n=3$ and $N=5$: 7, 2, 26; 7, 14, 22; 23, 2, 14; 23, 10, 10; 25, 2, 10.

In nest 4, where $n=4$ and $N=14$:

1, 28, 76; 1, 44, 68; 17, 56, 56; 23, 16, 76; 23, 44, 64; 41, 16, 68; 41, 28, 64; 47, 16, 64; 49, 8, 64; 49, 32, 56; 55, 20, 56; 55, 40, 44; 65, 20, 44; 79, 8, 16.

In nest 5, where $n=5$ and $N=41$:

1, 22, 242; 17, 14, 242; 17, 46, 238; 17, 106, 218; 17, 134, 202; 31, 38, 238; 31, 158, 182; 47, 14, 238; 47, 154, 182; 49, 2, 238; 49, 158, 178; 95, 118, 190; 95, 50, 218; 95, 130, 182; 97, 46, 218; 97, 62, 214; 97, 94, 202; 97, 134, 178; 113, 22, 214; 113, 62, 206; 113, 74, 202; 113, 146, 158; 127, 22, 206; 127, 46, 202; 127, 106, 178; 127, 134, 158; 143, 50, 190; 143, 74, 182; 143, 122, 154; 145, 70, 182; 161, 2, 182; 161, 38, 178; 193, 22, 146; 193, 62, 134; 193, 70, 130; 209, 22, 122; 209, 38, 118; 223, 22, 94; 223, 62, 74; 239, 22, 38; 241, 22, 22.

In nest 6, where $n=6$ and $N=122$:

17, 136, 716; 17, 436, 584; 49, 236, 668; 49, 224, 692; 49, 128, 716; 49, 496, 532; 55, 196, 700; 73, 116, 716; 73, 44, 724; 79, 32, 724; 79, 172, 704; 79, 112, 716; 79, 308, 656; 79, 340, 640; 79, 460, 560; 89, 224, 688; 89, 416, 592; 103, 376, 616; 103, 424, 584; 119, 68, 716; 119, 196, 692; 119, 436, 572; 119, 484, 532; 137, 4, 716; 137, 236, 676; 143, 124, 704; 143, 284, 656; 175, 196, 680; 175, 104, 700; 185, 40, 704; 185, 296, 640; 199, 136, 688; 199, 248, 656; 199, 304, 632; 199, 376, 592; 239, 308, 616; 241, 4, 688; 241, 416, 548; 241, 128, 676; 241, 464, 508; 247, 116, 676; 247, 236, 644; 271, 32, 676; 271, 220, 640; 271, 208, 644; 271, 380, 560; 313, 56, 656; 313, 304, 584; 329, 92, 644; 329, 460, 460; 337, 56, 644; 337, 196, 616; 359, 56, 632; 359, 152, 616; 359, 248, 584; 359, 424, 472; 367, 236, 584; 367, 296, 556; 401, 172, 584; 401, 248, 556; 401, 296, 532; 401, 364, 488; 409, 376, 472; 409, 152, 584; 431, 68, 584; 431, 136, 572; 431, 296, 508; 431, 376, 452; 439, 172, 556; 439, 196, 548; 439, 236, 532; 439, 284, 508; 455, 104, 560; 455, 280, 496; 457, 116, 556; 457, 364, 436; 473, 304, 464; 497, 196, 496; 497, 224, 484; 521, 44, 508; 521, 100, 500; 521, 220, 460; 521, 236, 452; 521, 340, 380; 527, 196, 464; 527; 284, 516; 529, 40, 500; 529, 116, 488; 529, 200, 460; 529, 268, 424; 529, 288, 436; 529, 332, 376; 551, 112, 464; 551, 304, 368; 583, 136, 416; 583, 224, 376; 593, 4, 424; 593, 196, 376; 623, 44, 376; 623, 104, 364; 623, 236, 296; 625, 196, 320; 625, 220, 304; 631, 28, 364; 631, 196, 308; 649, 4, 332; 649, 124, 308; 649, 172, 284; 649, 196, 268; 655, 4, 320; 655, 100, 304; 655, 40, 296; 655, 104, 280; 689, 32, 236; 689, 116, 208; 695, 4, 220; 695, 100, 196; 719, 32, 116; 719, 44, 112; 721, 28, 104; 721, 40, 100; 721, 56, 92.

In order to obtain the multiple sets in the several nests, we multiply the prime sets of all the preceding nests by the necessary power of 3.

The multiple sets in nest 4 are 3^3 times the prime set of nest 1, 3^2 times the prime sets of nest 2, and 3 times those of nest 3, making 8 in all.

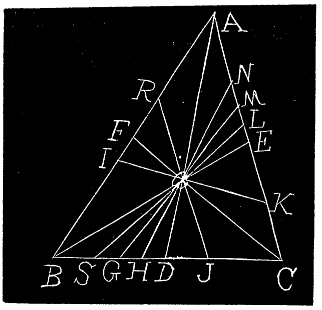
Also solved by *CHARLES C. CROSS*.

AVERAGE AND PROBABILITY.

56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.



Let ABC be the given triangle, D, E, F the mid-points of the sides, and O the center of gravity. Also let $DG=x$, DS, DH, DJ, FK, FL, FN, ER, or $EI=y$.

Now $FM=BM-\frac{1}{2}c=\dfrac{c(a-2x)}{2(a+6x)}$.

When the first point is on OG and the second on OH, OS, OJ, OK, OL, ON, OR, or OI, respectively, then the third point must fall on

$$MON=\frac{\triangle}{3}\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right),\ MOL=\frac{\triangle}{3}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right),$$

$$MARO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right),\ MAIO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right),$$

$$MACSO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1+\frac{4y}{c+6y}\right),\ MBHO=\frac{\triangle}{3}\left(2-\frac{4x}{a+6x}-\frac{4y}{c+6y}\right),$$

$$MOJB=\frac{\triangle}{3}\left(\frac{4y}{b+6y}+1-\frac{4x}{a+6x}\right),\ MOK=\frac{\triangle}{3}\left(1-\frac{4y}{b+6y}-\frac{4x}{a+6x}\right),$$

respectively.

The chance that the first is on OG is $dx/3a$; that the second is on OH, OS, or OJ is $dy/3a$; that it is on OK, OL, or ON is $dy/3c$; that it is on OR, or OI is $dy/3b$.

$$\therefore\ p=\frac{1}{27a^2}\left[\int_0^{\frac{1}{2}a}\left\{\int_0^x\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right)dy+\int_x^{\frac{1}{2}a}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right)dy\right\}dx\right.$$

$$\left.+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right)dxdy\right]+\frac{1}{27ac}\left[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right)dxdy\right.$$

$$\left.+\int_0^{\frac{1}{2}a}\left\{\int_0^{FM}\left(\frac{4x}{a+6x}+1+\frac{4y}{c+6y}\right)dy+\int_{FM}^{\frac{1}{2}c}\left(2-\frac{4x}{a+6x}-\frac{4y}{c+6y}\right)dy\right\}dx\right]$$

$$+\frac{1}{27ab}\left[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}b}\left(\frac{4y}{b+6y}+1-\frac{4x}{a+6x}\right)dxdy+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}b}\left(1-\frac{4y}{b+6y}-\frac{4x}{a+6x}\right)dxdy\right]$$

$$\therefore\ p=\frac{1}{27a}\int_0^{\frac{1}{2}a}\left[\tfrac{5}{2}+\frac{a-34x}{6(a+6x)}+\frac{4x(a-2x)}{(a+6x)^2}+\tfrac{4}{9}\log\left(\frac{a+6x}{a}\right)-\tfrac{2}{9}\log 4\right]dx$$

$$=\tfrac{1}{27}\left(\tfrac{1}{3}+\tfrac{5}{9}\log 4\right).$$

Since the first point can fall in any of the six portions into which the medians divide the triangle, the required probability is $P=6p$.

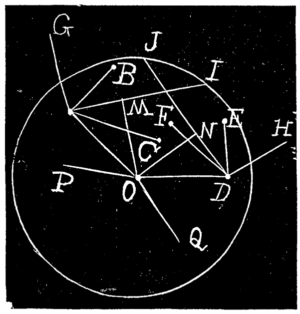
$$\therefore P=\tfrac{6}{27}(\tfrac{1}{3}+\tfrac{5}{9}\log 4)=\tfrac{1}{27}(2+\tfrac{10}{3}\log 4=\tfrac{2}{27}(1+\tfrac{10}{3}\log 2).$$

91. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

Six points A, B, C, D, E, F, are taken at random on the surface of a sphere. Find the chance that the plane through A, B, C intersects the plane through D, E, F within the sphere.

Solution by the PROPOSER.

Let AI, DJ be the diameters of the sections of the sphere made by the planes through A, B, C and D, E, F; M and N their centers, O the center of the sphere, OP a line such that AB is parallel to the plane MOP, OQ a line such that DE is parallel to the plane NOQ. Draw AG, DH perpendicular to AI, DJ, respectively.



Let $AO=r$, $\angle AOM=\theta$, $\angle DON=\varphi$, $\angle GAC=\psi$, $\angle GAB=\delta$, $\angle HDF=\mu$, $\angle HDE=\nu$, $\angle MOP=\lambda$, the angle the plane MOP makes with some fixed plane through $OP=\rho$, $\angle NOQ=\gamma$, and the diedral angle $MOQN=\eta$.

An element of surface at A is $4\pi r^2\sin\theta d\theta$; at D, $4\pi r^2\sin\varphi d\varphi$; at C, $4r^2\sin\theta\sin\psi d\psi d\lambda$; at B, $4r^2\sin\theta\sin(\psi-\delta)\sin\lambda\sin\delta d\delta d\rho$; at F, $4r^2\sin\varphi\sin\mu d\mu d\gamma$; at E, $4r^2\sin\varphi\sin(\mu-\nu)\sin\gamma\sin\nu d\nu d\eta$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and $\frac{1}{2}\pi$; of ψ, 0 and π; of δ, 0 and ψ; of μ, 0 and π; of ν, 0 and μ; of λ, 0 and π; of γ, $\pm(\theta-\varphi)$ and $\theta+\varphi$ (the double sign being taken $+$ when $\theta>\varphi$, and $-$ when $\theta<\varphi$); of ρ, 0 and 2π; of η, 0 and 2π.

Since the whole number of ways six points can be taken in the surface of the sphere is $(4\pi r^2)^6$ the required chance is

$$p=\frac{1}{(4\pi r^2)^6}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\int_0^{2\pi}\int_0^{2\pi} 4\pi r^2\sin\theta.d\theta 4\pi r^2$$

$$\sin\varphi d\varphi.4r^2\sin\theta\sin\psi d\psi d\lambda.4r^2\sin\varphi\sin\mu d\mu d\gamma.4r^2\sin\theta\sin(\psi-\delta)\sin\lambda\sin\delta d\delta d\rho$$

$$\times 4r^2\sin\varphi\sin(\mu-\nu)\sin\gamma\sin\nu d\nu d\gamma$$

$$=\frac{2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\int_0^{2\pi}\sin^3\theta\sin^3\varphi\sin\psi\sin\mu\sin(\psi-\delta)$$

$$\sin(\mu-\nu)\sin\delta\sin\nu\sin\lambda\sin\gamma d\theta d\varphi d\psi d\mu d\delta d\nu d\gamma d\rho$$

$$=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\sin^3\theta\sin^3\varphi\sin\psi\sin\mu\sin(\psi-\delta)$$

$\sin(\mu-\nu)\sin\delta\sin\nu\sin\lambda\sin\gamma\, d\theta d\varphi d\psi d\mu d\delta d\nu d\lambda d\gamma$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu\sin(\psi-\delta)\sin(\mu-\nu)$$

$\sin\delta\sin\nu\sin\lambda\, d\theta d\varphi d\psi d\mu d\delta d\nu d\lambda$

$$=\frac{16}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu\sin(\psi-\delta)\sin(\mu-\nu)\sin\delta\sin\nu$$

$d\theta d\varphi d\psi d\mu d\delta d\nu$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu(\sin\mu-\mu\cos\mu)\sin(\psi-\delta)\sin\delta$$

$d\delta d\varphi d\psi d\mu d\delta$

$$=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi(\sin\psi-\psi\cos\psi)\sin\mu(\sin\mu-\mu\cos\mu)$$

$d\theta d\varphi d\psi d\mu$

$$=\frac{3}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi(\sin\psi-\psi\cos\psi)\,d\theta d\varphi d\psi$$

$$=\tfrac{9}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^4\theta\sin^4\varphi\, d\theta d\varphi=\frac{27\pi}{64}\int_0^{\frac{1}{2}\pi}\sin^4\theta\, d\theta=\frac{81\pi^2}{1024}=(\tfrac{3}{4})^4(\tfrac{1}{2}\pi)^2.$$

NOTE. We may publish a second solution of Problem 90 in the next issue of the MONTHLY. ED. F.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

140. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

$\frac{1}{7}=0.\dot{1}4285\dot{7}$; $\frac{1}{14}=0.0\dot{7}1428\dot{5}$; $\frac{1}{21}=0.0\dot{0}4761\dot{9}$. Notice that the sum of the figures in each period is equal to 27. This is not true with $\frac{1}{72}$, $\frac{1}{73}$. Is there any general law of which these are special cases, and if so, what is it?

141. Proposed by **F. P. MATZ. M. Sc., Ph. D.**, Professor of Mathematics and Astronomy in Irving College, Mechanicsburg. Pa.

If the alloy in a half-dollar be 1-13th of the mass, and the coin be worth a cent if it be all alloy, what should be the exact value of the coin if it be all pure silver?

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

ALGEBRA.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Prove that

$$\begin{vmatrix} 1-\binom{m-1}{0} & -\binom{m-1}{1} & \cdots & \binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} \cdots & \binom{m-2}{m-3} \\ 1 & 0 & 1 \quad \cdots & \binom{m-3}{m-4} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 \quad 1 & \binom{1}{0} \\ 1 & 0 & 0 \ldots 0 & 1 \end{vmatrix} e=\sum_{n=\infty}^{n=1}\frac{n^m}{n!}, \text{ where } \binom{m-2}{k}=\frac{(n-2..(m-k-1)}{k!}.$$

130. Proposed by J. M. BOORMAN, Woodmere, N. Y.

Solve $x^5-y^5=2101....(1)$, $x-y=1....(2)$.

Find general formula for (1), ...(2), when $x^n-y^n=a$; for $n_0=3$; $n_1=5$; $n_2=7$; etc.

131. Proposed by HARRY S. VANDIVER. Bala, Montgomery County. Pa.

It is well known that, when we define the symbol $\sqrt[n]{a}$ after the manner of elementary text-books on algebra, certain *irrational equations* may be written down which have no real or imaginary roots. Required then, the condition, if any, between a, b, c, and d such that the equation, $ax+b+\sqrt{(cx^2+d)}=0$, shall have no root, real or or imaginary.

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

GEOMETRY.

146. Proposed by H. R. HIGLEY, M. Sc., Professor of Mathematics, Normal School. East Stroudsburg, Pa.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the line joining the points of concurrence=the sum of the squares on the two tangents from these points. Ex. 24, page 219, Mackay's *Elements of Euclid.*

147. Proposed by R. A. WELLS, Professor of Mathematics, Franklin College, New Athens, Ohio.

Find the locus in space of the point which is equally illuminated by each of two unequal lights whose intensities are a and b ($a>b$), placed at a distance c from each other.

160. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

Let GFH be the spherical triangle formed by joining the mid-points of the sides of the spherical triangle ABC; E the spherical excess of ABC; β, p the base and altitude of GFH. Prove $\sin\frac{1}{2}E=\sin\beta\sin p$.

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

CALCULUS.

123. Prize Problem. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{4}\pi}\log\tan\phi d\phi$.

A year's subscription to the MONTHLY will be given to the person sending to the Proposer the first solution of this problem. This problem was misstated in last issue.

124. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that the cardioides $r=a(1+\cos\theta)\dots(1)$, and $r=b(1-\cos\theta)\dots(2)$, intersect at right angles.

125. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Show that the *complete primitive* of the Differential Equation

$$\left[\tan^{-1}(x)-\frac{x}{1+x^2}\right]\frac{d^2y}{dx^2}=2\left(\frac{x}{(1+x^2)^2}\right)\left[x\frac{dy}{dx}-y\right],$$

is $y=C\tan^{-1}(x)+cx$.

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

EDITORIALS.

Dr. C. N. Little, professor of Mathematics in Leland Stanford University, has resigned his position.

Mr. S. W. Reaves, graduate scholar in Cornell University, has been appointed instructor in Mathematics at Orchard Lake Military Academy.

Professor Charles Hermite, the venerable dean of French mathematicians, died after a brief illness at his home in Paris, on January 14, 1901, and in his death the mathematical world sustains a great loss. He was born at Dieuze, December 25, 1822. In 1858, 1865, 1866, his transcendental solution of the quintic equation involving elliptic integrals was published in the *Comptes Rendus*. The theory of Differential Equations, the reduction of Abelian to Elliptic Functions, the Theory of Functions, and many other mathematical subjects, have received substantial additions at the hands of this great savant.

BOOKS AND PERIODICALS.

The Teaching of Mathematics in the Higher Schools of Prussia. By J. W. A. Young, Ph. D., Assistant Professor of the Pedagogy of Mathematics in the University of Chicago. 8vo. Cloth, 141 pages. Price, 80 cents. New York: Longmans, Green & Co.

The account of the Prussian High School System with detailed and specific description of the work in mathematics as set forth in this little volume is most timely—coming as it does at the close of the 19th century, the last part of the last twenty-five years of which has witnessed great changes and improved methods in mathematical teaching in America. There is still room for great improvement along the line and just at present great pressure is being brought to bear on mathematical teaching in colleges by the great universities, the colleges in turn are demanding a better quality of work in mathematics of the high school, and it is hoped that the result will be general improvement all along the line of mathematical teaching.

Dr. Young has gathered much of the material for the account respecting the Prussian Higher School System by personal observation, and in this account are to be found much that is of highest value to American teachers. All teachers in America who have been the subjects of political intrigue and personal whims rejoice to know that the German teacher works with a sense of security in his position without regard to political occurrences, or the whims of the powerful and influential, security in a modest compe-

tency while at work, security in his profession as a life work. This enables the German teacher to utilize all his energies in the improvement of his mind, and thus none are lost in the way of political scheming so common and so debasing in this country.

The American Journal of Mathematics. Edited by Frank Morley with the coöperation of Simon Newcomb, S. Cohen, Charlotte A. Scott and other mathematicians. Vol. XXIII, No. 1.

This number contains the following papers:

Die Typen der linearen Complexe rationaler Curven im *Rr.*, Von S. Kantor; Transformations of Systems of Linear Differential Equations, by E. J. Wilczynski; Distribution of the Ternary Linear Homogenious Substitutions in Galios Field into Complete Sets of Conjugate Substitutions, by L. E. Dickson; Distribution of the Quarternary Linear Homogenious Substitution in a Galois Field into Complete Sets of Conjugate Substitutions, by T. M. Putnam; On the Determination and Solution of the Metacyclic Quintic Equations with Rational Coefficients, by J. G. Glashan; Construction of the Geometry of Euclidean n-Dimensional Space by the Theory of Continuous Groups, by E. O. Lovett; A Table of Class Numbers for Cubic Number Fields, by L. W. Reid; On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity, by R. A. Roberts.

The Annals of Mathematics. Published in October, January, April, and July, under the auspices of Harvard University. Price, $2.00 per year in advance.

Among the articles in the January number for 1901 are the following: An Application of Elliptic Functions to Peaucellier's Link-Work, by Dr. Arnold Emch; Note on the Geometrical Treatment of Conics, by Professor Charlotte A. Scott; On a Special Class of Abelian Groups, by Dr. G. A. Miller; The Theory of Linear Dependence, by Maxime Bocher.

Divergent and Conditionally Convergent Series Whose Product is Absolutely Convergent. By Dr. Florian Cajori.

This is a reprint of a very interesting article which appeared in the Transactions of the American Mathematical Society.

ERRATA.

Pages 2—7, all square brackets should be parenthesis.

Pages 3—4, for H read K [kappa].

Page 4, first line, $(x-r)\lambda-1$ should read $((x-r)\lambda-1)$.

Page 5, in formula (14), $\binom{K+1}{\nu^1}$ should read $\binom{\nu^1}{K+1}$.

Page 7, line 6 from below $\begin{bmatrix}\nu^2\\ r\end{bmatrix}$ should be $\binom{\nu}{r}$.

Page 21, see the restatement of Problem 129, Algebra, in this number.

Page 22, see the restatement of Prize Problem, 123, Calculus, on page 54, of this number.

Page 24, line 5, for π read e.

Page 235, Vol. VII, Problem 103, Mechanics, should be numbered 109. Renumber accordingly, all the problems in Mechanics, proposed since then.

Problems 146 and 147, Geometry, in this issue, are proposed for the May number, Vol. VII.

THE AMERICAN MATHEMATICAL MONTHLY.

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VOL. VIII. MARCH, 1901. No. 3.

ON THE PRODUCT OF TWO COMMUTATIVE OPERATORS.

By DR. G. A. MILLER, Cornell University, Ithaca, New York.

The aim of this note is to give a very elementary explanation of the order of the product of two commutative operators. Such an explanation seems the more desirable on account of the fact that an error in regard to this matter occurs in one of the best known works on the theory of groups.* We shall first consider the cases when the orders of the two operators (s_1, s_2) are powers of the same prime number (p), the order of s_1 being p^α and that of s_2 being p^β ($\alpha \geqq \beta$).

From the equation $(s_1 s_2)^n = s_1^n s_2^n$ and the fact that $s_1^n s_2^n = 1$ only when $s_1^n = s_2^{-n}$, it follows that $s_1 s_2$ is of the same order as s_1 whenever $\alpha > \beta$. When $\alpha = \beta$ several cases present themselves: (1) The groups generated by s_1 and s_2 have only identity in common. In this case it follows from the given equations that $s_1 s_2$ is of the same order as s_1, viz., of order p^α. (2) s_1 is a power of s_2; *e. g.* $s_1 = s_2^\gamma$. In this case the order of $s_1 s_2$ may be any power of p from p^0 to p^α when $p > 2$, and p^0 to $p^{\alpha-1}$ when $p=2$. These orders may be obtained by assigning the following values to γ:

$$p^\alpha - 1,\ p^{\alpha-1} - 1,\ p^{\alpha-2} - 1,\ \ldots . p-1,\ 1.$$

In general, let $s_1 p^{\alpha_1}$ be the first power of s_1 which is also a power of s_2, so that $s_1 p^{\alpha_1} = s_2 k p^{\alpha_1}$ (α_1 does not equal α, and k being prime to p and less than $p^{\alpha - \alpha_1}$). It follows from the first sentence of the preceding paragraph that the

*Burnside, *Theory of Groups of a Finite Order*, 1897, page 16.

order of $s_1 s_2$ cannot be less than p^{α_1} nor greater than p^{α}. If we assign to k the following values:

$$p^{\alpha-\alpha_1}-1,\ p^{\alpha-\alpha_1-1}-1,\ p^{\alpha-\alpha_1-2}-1,\ \ldots . p-1,\ 1\ (\alpha_1 \text{ does not equal } \alpha),$$

we observe that $s_1 s_2$ may have for its order any power of p from p^{α_1} to p^{α} when p is odd (or when $p=2$ and $\alpha=\alpha_1$), and from p^{α_1} to $p^{\alpha-1}$ when $p=2$ and $\alpha>\alpha_1$.

These results are expressed by the following

THEOREM. *It is possible to find two commutative operators* (s_1, s_2) *of the same order* (p^{α}) *such that* $s_1 p^{\alpha_1}=s_2 k p^{\alpha_1}$ $(\alpha\overline{\gtrless}\alpha)$ *and* $s_1 s_2$ *is of order* p^{δ}, *where* δ *can have any value from* α_1 *to* α *when* $p>2$ *(or when* $p=2$ *and* $\alpha=\alpha_1$*), and from* α_1 *to* $\alpha-1$ *when* $p=2$ and $\alpha>\alpha_1$.

When the orders of s_1 and s_2 are not powers of the same prime they may be represented by $2^{\alpha_0} p_1{}^{\alpha_1} p_2{}^{\alpha_2} \ldots .$ and $2^{\beta_0} p_1{}^{\beta_1} p_2{}^{\beta_2} \ldots .$ respectively; $p_1, p_2, \ldots .$ being odd prime numbers, and the exponents being positive integers including 0. We may think of s_1 and s_2 as the products of operators of orders 2^{α_0}, $p_1{}^{\alpha_1}$, $p_2{}^{\alpha_2}$, ... and of orders 2^{β_0}, $p_1{}^{\beta_1}$, $p_2{}^{\beta_2}$, ... respectively. $s_1 s_2$ is then the product of all of these operators. Combining the pairs which are powers of the same prime, $s_1 s_2$ may be represented as the product of operators of orders 2^{γ_0}, $p_1{}^{\gamma_1}$, $p_2{}^{\gamma_2}$, ... ($p_x{}^{\gamma_x}$ being the product of the given operators of orders $p_x{}^{\alpha_x}$ and $p_x{}^{\beta_x}$). Since the groups generated by these operators have only identity in common $s_1 s_2$ is of order $2^{\gamma_0} p_1{}^{\gamma_1} p_2{}^{\gamma_2} \ldots .$ We have now reduced this case to one of the earlier ones.

It was observed in the second paragraph that γ_x is equal to the larger of the two numbers α_x and β_x whenever these are different, and that γ_x may be 0 whenever $\alpha_x=\beta_x$. *Hence the minimum order* (m) *of* $s_1 s_2$ *is the product of the highest of the primes which divide one and only one of their orders.** The maximum order (M) of $s_1 s_2$ is clearly the lowest common multiple of their orders. From the given theorem it follows that s_1 and s_2 can be so selected that the order of $s_1 s_2$ is any factor of M which is not less than m.

HYDRAULIC SOLUTION OF AN ALGEBRAIC EQUATION OF THE nth DEGREE.

By DR. ARNOLD EMCH, University of Colorado.

In the January (1901) number of this MONTHLY I have established two methods of extracting the nth root of any positive real number. In conclusion, I proposed to apply the first (hydrostatic) method to the solution of an equation of the form

$$a_0+a_1 x+a_2 x^2+\ldots .+a_n x^n=0 \ldots .(1).$$

*Mr. Fite first called my attention to this minimum value.

It is as follows: Take n solids of revolution corresponding to the volumes x, x^2, $x^3 \dots x^n$ and attach them to a lever which may turn about a fixed edge H. The axes of these solids must be kept in a vertical position. Let a_1, a_2, $a_3 \dots a_n$ be the distances from H of the points where these axes intersect the lever. Attach also a weight 1 to the lever at a distance $-a_0$ from H. According as a_k is positive or negative attach the corresponding solid to the right or left of H. The lowest points of the solids are all in the same straight line. The whole apparatus is now put in a vessel containing water and the level of water is raised until the lever and with it the lowest points of the solids are all in a horizontal position. In this position the depths of immersion of the solids are all equal to a certain length y and the quantities of water displaced are y, y^2, $y^3 \dots y^n$. These act upon the lever as vertical forces in an upward direction.

The moment of all these forces is therefore

$$a_0+a_1y+a_2y^2+\dots+a_ny^n \dots (2).$$

The other forces acting upon the lever are the weights of the solids themselves. If these are all equal to w, their moment is

$$w(a_1+a_2+a_3+\dots+a_n), \text{ hence}$$

$$a_0+a_1y+a_2y^2+\dots+a_ny^n+w(a_1+a_2+a_3+\dots+a_n)=0 \dots (3).$$

Adding now a weight w at a distance $-(a_1+a_2+a_3+\dots+a_n)$ from H and adjusting the water-level in such a manner that the lever assumes again a horizontal position, then the depth of immersion will be x, since in this case

$$a_0+a_1x+a_2x^2+\dots+a_nx^n=0 \dots (4).$$

Hence, the depth of immersion of the solids is a positive real root of (1).

After the first root has been found, the water-level in the vessel may be raised still higher. Equilibrium will first be destroyed, but at a certain point of the level, equilibrium is again restored and the corresponding depth of immersion will give another positive real root of (1). By this process all positive real roots of the proposed equation may be obtained.

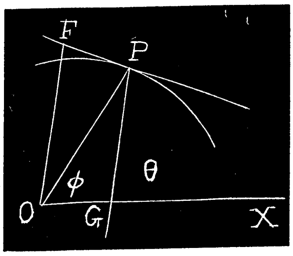
I shall not enter into further details of this interesting problem. Since the publication of my first article on this subject I have found that M. Georges Meslin has published a similar method in the *Journal de Physique* for June, 1900, Vol. IX, III series, page 339. The reader is referred to Professor Meslin's valuable article, where a description of an apparatus for the solution of equations (1) may be found.

Boulder, Col., February 9, 1901.

THE LENGTH OF A DEGREE OF LATITUDE AND LONGITUDE FOR ANY PLACE.

By GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the earth, P any place on the surface, PG the normal and PF the tangent at P, $OP=p=$the perpendicular from the center on the tangent. a, b the semi-axes of the earth, $\rho=$radius of curvature at P, $OP=r$ $=$the radius vector, $\angle POX=\varphi$, $\angle PGX=FOX=\theta$, $l=$length of a degree of meridian, $\rho'=$radius of circle of latitude, $L=$length of degree of latitude, $e^2=\dfrac{a^2-b^2}{a^2}=.00680349=$square of the eccentricity, since $a=6378190$ meters$=3963.296$ miles.



Now $\rho=\dfrac{a^2b^2}{p^3}$, but $p=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=a\sqrt{1-e^2\sin^2\theta}$.

$$\therefore\ \rho=\frac{a(1-e^2)}{(1-e^2\sin^2\theta)^{\frac{3}{2}}},\quad l=\frac{2\pi\rho}{360}=\frac{\pi\rho}{180}.$$

$$\therefore\ l=\frac{\pi a(1-e^2)}{180(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{68.70175}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\text{ miles}=\frac{110562.7346}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\text{ meters}.$$

$$\frac{1}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=1+\tfrac{3}{2}e^2\sin^2\theta+\tfrac{15}{8}e^4\sin^4\theta+\tfrac{105}{48}e^6\sin^6\theta+\ldots.$$

The fourth term can be omitted as its greatest value will not affect the result more than three inches.

$$\sin^2\theta=\tfrac{1}{2}(1-\cos2\theta),\quad \sin^4\theta=\tfrac{1}{8}(3-4\cos2\theta+\cos4\theta).$$

These values, with the value of e, give

$$\frac{1}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=1.00514058-.00515323\cos2\theta+.00001265\cos4\theta.$$

$$\therefore\ l=69.054917-.354036\cos2\theta+.000869\cos4\theta\text{ miles}$$

$$=111131.09118-569.75520\cos2\theta+1.39862\cos4\theta\text{ meters}.$$

$$\rho'=r\cos\varphi=\frac{a\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$L = \frac{2\pi\rho'}{360} = \frac{\pi\rho'}{180} = \frac{\pi a \cos\theta}{180\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\ L = \frac{69.1726\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}\ \text{miles} = \frac{111320.4635\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}\ \text{meters}.$$

$$\frac{1}{\sqrt{(1-e^2\sin^2\theta)}} = 1+\tfrac{1}{2}e^2\sin^2\theta+\tfrac{3}{8}e^4\sin^4\theta+\tfrac{15}{48}e^6\sin^6\theta+\ldots.$$

$$=(1+\tfrac{1}{2}e^2+\tfrac{3}{8}e^4)-(\tfrac{1}{2}e^2+\tfrac{3}{4}e^4)\cos^2\theta+\tfrac{3}{8}e^4\cos^4\theta\ldots.$$

$$\cos^3\theta=\tfrac{1}{4}(\cos 3\theta+3\cos\theta),\quad \cos^5\theta=\tfrac{1}{16}(\cos 5\theta+5\cos 3\theta+10\cos\theta).$$

$$\therefore\ L=111320.4635[(1+\tfrac{1}{8}e^2+\tfrac{3}{64}e^4)\cos\theta-(\tfrac{1}{8}e^2+\tfrac{9}{128}e^4)\cos 3\theta+\tfrac{2}{128}e^4\cos 5\theta]$$

$$=111415.37533\cos\theta-95.03428\cos 3\theta+.12022\cos 5\theta.$$

Since the greatest value of the last term is not over five inches it can be omitted.

$$\therefore\ L=111415.37533\cos\theta-95.03428\cos 3\theta\ \text{meters}$$

$$=69.23155\cos\theta-.05905\cos 3\theta\ \text{miles}.$$

The following table gives the length of a degree at intervals of five degrees.

	l		L	
Degrees	*Meters*	*Miles*	*Meters*	*Miles*
0°	110562.7346	68.70175	111320.3411	69.17250
5°	110571.305	68.70701	111051.725	69.00558
10°	110596.769	68.72288	109640.673	68.12878
15°	110638.365	68.74873	107552.254	66.83108
20°	110694.879	69.78386	104648.397	65.02667
25°	110764.615	68.82719	100952.272	62-72996
30°	110845.514	68.87746	96489.058	59.95660
35°	110935.152	68.93315	91290.501	56.72631
40°	111030.839	68.99261	85396.151	53.06366
45°	111129.693	69.05405	78850.126	48.99608
50°	111228.715	69.11556	71698.992	44.55249
55°	111324.887	69.17532	63997.427	39.96687
60°	111415.270	69.23150	55802.722	34.67483
65°	111497.081	69.28234	47178.162	29.31568
70°	111567.789	69.32627	38188.589	23.72972
75°	111625.216	69.36194	28903.727	17.96027
80°	111667.556	69.38825	19394.797	12.05159
85°	111693.506	69.40438	9735.561	6.04951
90°	111702.245	69.40988	0.000	0.000

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

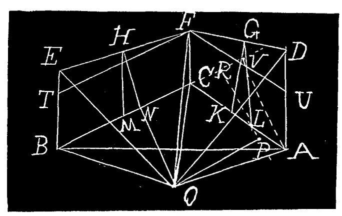
ARITHMETIC.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

At the corners of a triangle sides a, b, c feet, are towers d, e, f feet high. At what point must a ladder be placed so that it will just reach to the top of each tower without moving? How long is the ladder? Substitute $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$; d, e, f at A, B, C, respectively.

Solution by the PROPOSER.

Let ABC be the triangle, $AD=d$, $BE=e$, $CF=f$, the towers. Join DF, EF, and draw UF parallel to AC, TF parallel to BC. From G, the mid-point of DF, draw GK parallel to AD, GL perpendicular to DF. From H, the mid-point of EF, draw HM parallel to BE, HN parallel to EF. Draw ON perpendicular to BC, and OL perpendicular to AC. Then O is the required foot of the ladder. For O is equally distant from D, E, F, since OL is perpendicular to the plane $ADFC$ at L, and ON is perpendicular to the plane $BCFE$ at N. Draw LR, AV perpendicular to BC, OP perpendicular to LR.



Then $DU=d-f$, $EF=e-f$, $GK=\frac{1}{2}(d+f)$, $HM=\frac{1}{2}(e+f)$.

In the similar triangles LGK and DFU, $LK:GK=DU:UF$.

$$\therefore\ LK=\frac{d^2-f^2}{2b},\quad CL=\tfrac{1}{2}b+\frac{d^2-f^2}{2b}=\frac{b^2+d^2-f^2}{2b}.$$

Similarly $MN=\dfrac{e^2-f^2}{2a}$, $CN=\frac{1}{2}a-\dfrac{e^2-f^2}{2a}=\dfrac{a^2+f^2-e^2}{2a}$.

$AV=\dfrac{2\triangle}{a}$, where $\triangle$=area ABC. $VC=\sqrt{\dfrac{b^2-4\triangle^2}{a^2}}$.

$$\therefore\ VC=\frac{a^2+b^2-c^2}{2a}.\quad RC:LC=VC:AC.$$

$$\therefore\ RC=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)}{4ab^2}.$$

$$RN=OP=RC+NC=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)+2b^2(a^2+f^2-e^2)}{4ab^2}.$$

$OL:OP=AC:AV$.

$$\therefore\ OL=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)+2b^2(a^2+f^2-e^2)}{8\triangle b}.$$

$OF=\sqrt{(OL^2+CL^2+CF^2)}$.

When $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$,

$OL^2=60363.9509$, $CL^2=9506.25$, $CF^2=900$.

$\therefore\ OF=266.03$ feet.

Also solved by *J. SCHEFFER.*

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

"A pound of gold may be drawn into a wire that would extend around the earth." What would be the diameter of such a wire if the specific gravity of gold is 19.36 and the distance is 24,900 miles?

Solution by J. M. ARNOLD, Crompton, R. I.; G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; and J. SCHEFFER, A. M., Hagerstown, Md.

62.4 pounds=weight of 1 cubic foot of water. Then, the specific gravity of gold being 19.36, the weight of 1 cubic foot of gold is 19.36×62.4 pounds, or 1208.064 pounds.

Hence, in 1 pound of gold there are $\frac{1728}{1208.64}$ or 1.43039 cu. inches nearly.

$\therefore\ \frac{1}{4}\pi d^2\times24900\times5280\times12=1.43039$ cubic inches.

$\therefore\ d=.000034$ inches, nearly.

Mr. Arnold remarks that to measure so small a quantity one would have to estimate 1-12 of one of the divisions of a Brown and Sharp's Micrometer Gage, which reads to the hundredth of a millimeter.

Also solved by *ELMER SCHUYLER.*

139. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

The *ratio* of the interest to the true discount on a certain principal for a certain time at a certain rate per cent. per annum, is $m=21$ to $n=20$. What is the rate per cent.?

Solution by P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and ELMER SCHUYLER, M. Sc., Professor of Mathematics, Boys' High School, Reading, Pa.

Let P be the principal ; r, the rate ; and t, the time in years.

Then the interest, I, is trP.

The true discount$=\frac{trP}{1+rt}$. $\therefore\ trP:\frac{trP}{1+tr}=m:n=21:20$.

$\therefore\ 1+rt=m/n=\frac{21}{20}$, and $rt=\frac{m-n}{n}=\frac{1}{20}$, or $r=\frac{1}{20t}$.

Thus r depends on the time.

If $t=1$ year, $r=5\%$.

Also solved by *G. B. M. ZERR*, *J. M. ARNOLD.* and *J. SCHEFFER.*

ALGEBRA.

113. Proposd by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find a square number consisting of 24 figures such that the numbers formed by the first 12 figures and the last 12 figures, respectively, are consecutive, and *vice versa*.

Solution by the PROPOSER.

(1). Let $y^2=1000000000001x+1000000000000$ be the number.

$\therefore 1000000000001x=y^2-1000000000000$.

$\therefore\ x=\dfrac{(y+1000000)(y-1000000)}{10001\times 99990001}$.

Let $y+1000000=99990001u$, $y-1000000=10001v$.

$\therefore 10001v=99990001u-2000000.\quad v=9998u-199+\dfrac{3u-9801}{10001}$.

Let $S=\dfrac{3u-9801}{10001}$. $\therefore\ u=\dfrac{10001S+9801}{3}=3333S+3267+\dfrac{2S}{3}$.

Let $S=3m$. $\therefore\ u=10001m+3267$.

$y=1000000000001m+326666333267$.

$x=(99990001m+32663267)(10001m+3267)$.

Let $m=0$. $\therefore\ y=326666333267$, $x=106710893289$.

$y^2=106710893290106710893289=1000000000000(x+1)+x$.

(2). $y^2=1000000000001x+1$, or $x=\dfrac{(y+1)(y-1)}{(10001)(99990001)}$.

$y+1=99990001u$, $y-1=10001v$.

$\therefore\ v=\dfrac{9998u+(3u-2)}{10001}=9998u+S$. $\therefore\ u=\dfrac{3333S+(2S+2)}{3}=3333S+t$.

$\therefore\ S=t-1+\frac{1}{2}t=t-1+m$. $\therefore\ S=3m-1$, $u=10001m-3333$.

$y=1000000000001m-333266673334$.

$x=(10001m-3333)(99990001m-33323335)$.

Let $m=1$.

$\therefore\ y=666733326667$, $x=444533328888$.

$y^2=444533328888444533328889=1000000000000x+(x+1)$.

114. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

$$\left.\begin{array}{l} bx^3=10a^2bx+3a^3y \\ ay^3=10b^2ay+3b^3x \end{array}\right\}$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

$bx^3=10a^2bx+3a^3y\ldots\ldots(1)$, $ay^3=10ab^2y+3b^3x\ldots\ldots(2)$.

Let $x/a=u$, $y/b=v$, then (1) and (2) become

$u^3=10u+3v\ldots\ldots(3)$, $v^3=10v+u\ldots\ldots(4)$.

(3)+(4) gives $u^3+v^3-13(u+v)=0\ldots\ldots(5)$.

(3)−(4) gives $u^3-v^3-7(u-v)=0\ldots\ldots(6)$.

From (5), $u+v=0\ldots\ldots(7)$; $u^2-uv+v^2=13\ldots\ldots(8)$.

From (6), $u-v=0\ldots\ldots(9)$, $u^2+uv+v^2=7\ldots\ldots(10)$.

From (7) and (9), $u=0$, $v=0$.

(8)+(10) gives $u^2+v^2=10\ \ldots\ldots(11)$.

(8)−(10) gives $uv=-3\ldots\ldots(12)$.

From (11) and (12), $u+v=\pm2$, $u-v=\pm4$.

$\therefore\ u=\pm3$ or ±1, $v\mp1$ or ∓3.

From (7) and (10), $u=\pm\sqrt{7}$, $v=\mp\sqrt{7}$.

From (9) and (8), $u=\pm\sqrt{13}$, $v=\pm\sqrt{13}$.

$\therefore\ u=0,\ 1,\ -1,\ 3,\ -3,\ \sqrt{7},\ -\sqrt{7},\ \sqrt{13},\ -\sqrt{13}$.

$v=0,\ -3,\ 3,\ -1,\ 1,\ -\sqrt{7},\ \sqrt{7},\ \sqrt{13},\ -\sqrt{13}$.

$x=0,\ a,\ -a,\ 3a,\ -3a,\ a\sqrt{7},\ -a\sqrt{7},\ a\sqrt{13},\ -a\sqrt{13}$.

$y=0,\ -3b,\ 3b,\ -b,\ b,\ -b\sqrt{7},\ b\sqrt{7},\ b\sqrt{13},\ -b\sqrt{13}$.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let $bx/ay=t$.

Divide (1) by (2), then $\dfrac{b^3x^3}{a^3y^3}=\dfrac{10bx+3ay}{10ay+3bx}$, or $t^3=\dfrac{10t+3}{10+3t}$.

Whence, $3t^4+10t^3-10t-3=0$.

Factoring, $(t^2-1)(3t^2+10t+3)=0$.

Whence, $t=1,\ -1,\ -\frac{1}{3},\ -3$.

When $t=1$, $bx=ay$ and substituting in (1) we easily get,

$$x=0, \text{ or } \pm a\sqrt{13}.$$
$$y=0, \text{ or } \pm b\sqrt{13}.$$

When $t=-1$, $bx=-ay$. and we get in a similar manner,

$$x=0, \text{ or } \pm a\sqrt{7}.$$
$$y=0, \text{ or } \mp b\sqrt{7}.$$

When $t=-\frac{1}{3}$, $\quad x=0$, or $\pm3a$.

$y=0$, or $\mp b$.

When $t=-3$, $x=0$, or $\pm a$.
$y=0$, or $\mp 3b$.

Also solved by *J. M. BOORMAN*, and *HARRY S. VANDIVER.*

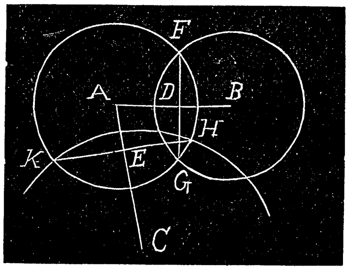
GEOMETRY.

143. Proposed by J. T. FAIRCHILD, A. M., Instructor in Mathematics, Crawfis College, Crawfis College, Ohio.

If the centers of three spheres do not lie in the same straight line, their surfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centers and equal distances from this plane on opposite sides. [From *Phillips and Fisher's Geometry.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B, C be the centers of the three spheres, respectively. B intersects A in the circle, radius FD, such that FD is perpendicular to AB and D lies on AB. C intersects A in the circle, radius EH, such that EH is perpendicular to AC and E lies on AC. The two circles D and E on the same sphere A can intersect in only two points situated on a line perpendicular to the plane AED and equally distant from it. But plane AED coincides with plane ABC. Therefore, the truth of the theorem follows.



144. Proposed by L. C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.

Find the equations of four cones that pass through three given straight lines intersecting in the same point.

I. Solution by the PROPOSER.

Let the mutual inclination of the line be 2α, 2β, 2γ, and let the equation of the cone be referred to the three given straight lines as coördinate axes.

The equation of the concentric sphere referred to the same axes is

$$x^2+y^2+z^2+2yz\cos2\alpha+2zx\cos2\beta+2xy\cos2\gamma=r \quad \ldots(1).$$

The equation of the plane that passes through the intersection of the cone and (1) is

$$\frac{x}{r}+\frac{y}{r}+\frac{z}{r}=1\ldots.(2).$$

Now making (1) homogeneous by means of (2), and reducing, we have for the equation of the cone,

$$\frac{\sin^2\alpha}{x}+\frac{\sin^2\beta}{y}+\frac{\sin^2\gamma}{z}=0.$$

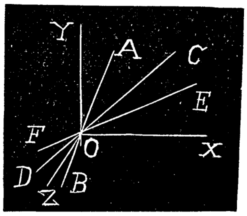
By using supplementary angles, the other equations are

$$\frac{\sin^2\alpha}{x}+\frac{\cos^2\beta}{y}+\frac{\cos^2\gamma}{z}=0,$$

$$-\frac{\cos^2\alpha}{x}+\frac{\sin^2\beta}{y}+\frac{\cos^2\gamma}{z}=0, \text{ and } -\frac{\cos^2\alpha}{x}-\frac{\cos^2\beta}{y}+\frac{\sin^2\gamma}{z}=0.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O, the intersection of the three lines, be the origin of rectangular co-ordinates, and AB, CD, EF the three lines,



$$\left.\begin{array}{l}x=az\\y=bz\end{array}\right\}\ldots.(1),\ \left.\begin{array}{l}x=cz\\y=dz\end{array}\right\}\ldots.(2),\ \left.\begin{array}{l}x=fz\\y=gz\end{array}\right\}\ldots.(3),$$

the equations of AB, CD, DE, respectively, and

$$\left.\begin{array}{l}x=tz\\y=uz\end{array}\right\}\ldots.(4),$$

the equation of the axes of the cone having AO, BO, CO in the surface of one nappe. Then

$$\left.\begin{array}{l}x=tz\\y=-uz\end{array}\right\}\ldots.(5),\ \left.\begin{array}{l}x=-tz\\y=uz\end{array}\right\}\ldots.(6),\ \left.\begin{array}{l}x=-tz\\y=-uz\end{array}\right\}\ldots.(7),$$

are the equations of the axes of the remaining three cones.

$$\therefore\ (x-tz)^2+(y-uz)^2=e^2z^2(1+t^2+u^2)\ldots.(8),$$

$$(x-tz)^2+(y+uz)^2=e_1^2z^2(1+t^2+u^2)\ldots.(9),$$

$$(x+tz)^2+(y-uz)^2=e_2^2z^2(1+t^2+u^2)\ldots.(10),$$

$$(x+tz)^2+(y+uz)^2=e_3^2z^2(1+t^2+u^2)\ldots.(11),$$

are the equations of the cone.

Since (4) must make equal angles with (1), (2), (3), we get at once, if θ is this angle,

$$\cos\theta=\frac{1+at+bu}{\sqrt{[1+a^2+b^2]}\sqrt{[1+t^2+u^2]}}=\frac{1+ct+du}{\sqrt{[1+c^2+d^2]}\sqrt{[1+t^2+u^2]}}$$

$$=\frac{1+ft+gu}{\sqrt{[1+f^2+g^2]}\sqrt{[1+t^2+u^2]}}.$$

$$\therefore \frac{1+at+bu}{\sqrt{[1+a^2+b^2]}}=\frac{1+ct+du}{\sqrt{[1+c^2+d^2]}}=\frac{1+ft+gu}{\sqrt{[1+f^2+g^2]}} \text{ determines } t \text{ and } u.$$

$$e^2=\tan^2\theta=\frac{(t-a)^2+(u-b)^2+(au-bt)^2}{(1+at+bu)^2},\quad e_1^2=\frac{(t-a)^2+(u+b)^2+(au+bt)^2}{(1+at-bu)^2},$$

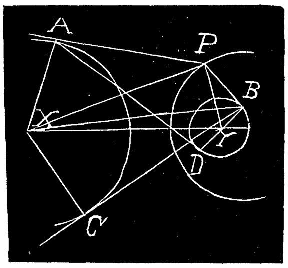
$$e_2^2=\frac{(t+a)^2+(u-b)^2+(au+bt)^2}{(1-at+bu)^2},\quad e_3^2=\frac{(t+a)^2+(u+b)^2+(au-bt)^2}{(1-at-bu)^2}.$$

145. Proposed by FRANK GIFFIN, Graduate Student, State University, Boulder, Col.

If A and B be the points of contact, upon two circles X and Y, of tangents drawn from any point of their circle of similitude, then the tangent from A to Y is equal to the tangent from B to X. [From *Casey's Sequel to Euclid*, Part I., page 144.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be any point on the circle of similitude, AP, BP the tangents from P to X and Y, respectively. AD, BC the tangents from A and B to Y and X, respectively.



Let $AX=R$, $BY=r$. Let $AD=a$, $BC=b$, $AP=c$, $BP=d$, $PX=m$, $PY=n$.

$\angle APX=\angle BPY$ since P is on circle of similitude. $\therefore$ $\angle APY=\angle BPX=\theta$.

Also, $c:d=R:r$. $\therefore$ $d=cr/R$....(1).

$m:n=R:r$. $\therefore$ $n=mr/R$....(2).

$a^2=AY^2-r^2=c^2+n^2-2cn\cos\theta-r^2$

$\quad=c^2+m^2r^2/R^2-(2cmr/R)\cos\theta-r^2$....(3).

$b^2=BX^2-R^2=d^2+m^2-2dm\cos\theta-R^2=c^2r^2/R^2+m^2-(2cmr/R)\cos\theta-R^2$..(4).

(3)−(4) gives $R^2(a^2-b^2)=(c^2-m^2+R^2)(R^2-r^2)$.

But $c^2+R^2=m^2$.

$\therefore$ $a^2-b^2=0$. $\therefore$ $a=b$.

CALCULUS.

106. Proposed by M. C. STEVENS, M. A., Professor of Higher Mathematics, Purdue University, Lafayette, Ind.

$$\int_0^\pi \frac{\cos rx\,dx}{1-2a\cos x+a^2}=\frac{\pi a^r}{1-a^2}.$$

[*Williamson's Integral Calculus*, 6th Edition, page 174.]

Solution by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

In Todhunter's *Plane Trigonometry*, 3d Edition, Art. 309, we have

$$\cos\alpha+a\cos(\alpha+\beta)+a^2\cos(\alpha+2\beta)+\ldots\ a^{n-1}\cos[\alpha+(n-1)\beta]$$

$$=\frac{\cos\alpha-a\cos(\alpha-\beta)}{1-2a\cos\beta+a^2}\ldots.(1),$$

if n be indefinitely increased. Make $\alpha=0$. Then

$$1+a\cos\beta+a^2\cos2\beta+\ldots.a^{n-1}\cos[(n-1)\beta]=\frac{1-a\cos\beta}{1-2a\cos\beta+a^2}\ldots.(2).$$

Multiply (2) by 2 and subtract a unit from both members of the resulting equation, then

$$\frac{1-a^2}{1-2a\cos\beta+a^2}=1+2a\cos\beta+2a^2\cos2\beta+\ldots.2a^{n-1}\cos[(n-1)\beta]\ldots.(3).$$

$$\text{Then, }\int_0^\pi\frac{\cos rx\,dx}{1+2a\cos x+a^2}=\int_0^\pi\frac{\cos rx}{1-a^2}\{1+2a\cos x+2a^2\cos2x+\ldots.$$

$$+2a^{n-1}\cos[(n-1)x]\}dx=\frac{2a^r}{1-a^2}\int_0^\pi\cos^2rx\,dx=\frac{\pi a^r}{1-a^2},$$

the terms of the series but this one vanishing between the given limits, as may be seen after reducing for a few terms.

Also solved by *G. B. M. ZERR*, *W. W. LANDIS*, *J. SCHEFFER*, and *L. C. PLANT*.

107. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

The speed of signaling in submarine telegraph-cable varies as $x^2\log(1/x)$, in which x is the ratio of the radius of the core to that of the covering. Prove that the *maximum speed* is attained when this ratio is $1:\sqrt{e}$.

I. Solution by J. W. YOUNG, Cornell University, Ithaca, N. Y.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

$$\text{If } y=x^2\log\left(\frac{1}{x}\right)=-x^2\log x,\quad \frac{dy}{dx}=-2x\log x-x=-x(\log x^2+1).$$

$\log x^2+1=0$ gives the maximum.

$$\therefore\ \log\frac{1}{x^2}=1.\quad\therefore\ \frac{1}{x^2}=e,\text{ and } x=\frac{1}{\sqrt{e}}.$$

II. Solution by H. C. WHITAKER, Ph.D., Manual Training School, Philadelphia, Pa.; W. W. LANDIS, A.M., Dickinson College, Carlisle, Pa.; J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Tex.; ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.; and J. JCHEFFER, A. M., Hagerstown, Md.

$$\text{If } S=cx^2\log\frac{1}{x},\quad \frac{dS}{dx}=2cx\log\frac{1}{x}-cx.$$

Dividing by cx and equating to zero, $\log\frac{1}{x}=\frac{1}{2}$.

Whence, $\frac{1}{x}=\sqrt{e}$, and $x=\frac{1}{\sqrt{e}}$.

MECHANICS.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridged is $p_1=80$ tons. The track is supposed to be straight and practically horizontal. Had the locomotive and tender been running at the rate of $r=60$ miles an hour, how many tons would the pressure on the bridge have been?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

Regarding the earth as a perfect sphere and neglecting its rotation, we get for latitude, θ.

Centrifugal force$=\frac{Wv^2}{gR\cos\theta}=f$,

where $W=80$ tons$=160000$ pounds, $v=60$ miles an hour$=88$ feet per second, $g=$gravity$=32.16$ feet, $R=$radius of earth$=3956$ miles$=20887680$ feet.

$\therefore f=\frac{1239040000}{671747788.8\cos\theta}=\frac{1.8445}{\cos\theta}$ pounds.

If the locomotive is running on a great circle, $f=1.8445$ pounds.

$P=$pressure$=W-f=159998.1555$ pounds.

$\therefore$ $P=79.99908$ tons, or practically 80 tons, the original weight.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

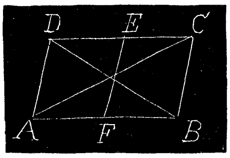
Let $2a$, $2b$ be the diagonals of a rhombus, φ the angle the principal axis at the mid-point of a side makes with the diagonals. Prove $\tan2\varphi+\frac{3}{5}\tan\beta$, when β is an angle of the rhombus. The principal moments of inertia about this mid-point of the side are $\frac{1}{24}m\{5a^2+5b^2\pm\sqrt{[25(a^2+b^2)^2-64a^2b^2]}\}$.

Solution by the PROPOSER.

Let $ABCD$ be the rhombus, side c. EF, FB the axes; $\angle DAB=\angle EFB=\beta$; $\angle\theta=$the angle the principal axis makes with the side AB at its mid-point F; $\angle\varphi$, the angle the principal axis makes with the diagonal.

$$\text{Then } \Sigma mxy=\rho\sin^2\beta\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c(x+y\cos\beta)ydxdy=\tfrac{1}{3}\rho c^2\sin\beta\cos\beta=\tfrac{1}{3}mc^2\sin\beta\cos\beta.$$

$$\Sigma mx^2=\rho\sin\beta\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c(x+y\cos\beta)^2dxdy=\tfrac{1}{12}m(c^2+4c^2\cos^2\beta).$$



$$\Sigma my^2=\rho\sin^3\beta\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c y^2\,dxdy=\tfrac{1}{3}mc^2\sin^2\beta.$$

$$\therefore\ \tan2\theta=\frac{4\sin2\beta}{1+4\cos2\beta}.$$

But $2\theta=\beta+2\varphi$. $\therefore\ \tan2\varphi=\frac{3}{5}\tan\beta$.

$A\cos^2\theta+\beta\sin^2\theta=\frac{1}{2}m(c^2+4c^2\cos^2\beta)$, $A\sin^2\theta+\beta\cos^2\theta=\frac{1}{3}mc^2\sin^2\beta$.

$\therefore\ A=\frac{1}{24}mc^2[5+\surd(17+8\cos2\beta)]$. $B=\frac{1}{24}mc^2[5-\surd(17+8\cos2\beta)]$.

$$c^2=a^2+b^2,\ \cos\beta=\frac{a^2-b^2}{a^2+b^2},\ \cos2\beta=\frac{(a^2+b^2)^2-8a^2b^2}{(a^2+b^2)^2}.$$

$\therefore\ A=\frac{1}{24}m\{5a^2+5b^2+\surd[25(a^2+b^2)-64a^2b^2]\}$.

$B=\frac{1}{24}m\{5a^2+b^2-\surd[25(a^2+b^2)-64a^2b^2]\}$.

106. Proposed by J. E. CRAIG, A. B., New Germantown, N. J.

The centers of the two wheels of a bicycle are three feet apart.

(1) If a rider wishes the rear wheel to trace a circle 14 feet in diameter, what must be the diameter of the circle traced by the front wheel?

(2) If the rider weighs 120 pounds, and his center of gravity is 3 feet from the ground, at what angle must he lean to make one revolution of the circle every 3 seconds?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) The line that is tangent to the hind wheel's path passes through the center of the front wheel, caused by the rigid frame work.

$\therefore\ R=\surd(9+49)=\surd58=7.615773$ feet.

$2R$=diameter traced by front wheel$=15.231546$ feet.

(2) Since the man must be in equilibrium we do not use his weight, and we may regard him as moving around in the plane of the track. Suppose he is midway between the wheels.

Let θ be his inclination to the vertical, g=gravity, $f=v^2/r$=centrifugal force. Then $g\sin\theta$=component of gravity perpendicular to man's direction, $f\cos\theta$=component of centrifugal force perpendicular to same direction. When in equilibrium these forces equalize each other.

$\therefore\ g\sin\theta=f\cos\theta=(v^2/r)\cos\theta$.

$\therefore\ \tan\theta=v^2/gr$.

$r=\surd(49+\frac{9}{4})=7.1589$ feet. $v=14.3178\pi/3$ feet$=14.9936$ feet per second.

$$\therefore\ \tan\theta=\frac{(14.9936)^2}{32\frac{1}{6}\times7.1589}=.97625.$$

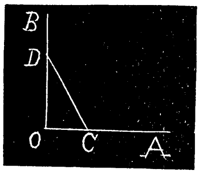
$\therefore\ \theta=44^\circ\ 18'\ 41''$.

107. Proposed by M. E. GRABER, Student, Heidelberg University, Tiffin, O.

Two particles attracting each other inversely as the square of their distances apart, are constrained to move in straight lines which intersect each other at right angles. How long will it take for the particles to meet and how far does each particle *move?*

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $OA=a$, $OB=b$, $OD=u$, $OC=x$, m=mass of particle at C, n=mass of particle at D; A, B the positions of the particles at the beginning of motion; C, D their positions at any time t; v, v_1 their velocities at C, D. Then the equations of motion are



for C, $\dfrac{d^2x}{dt^2}=\dfrac{nx}{(x^2+u^2)^{\frac32}}=F$, since $s=x$,

for D, $\dfrac{d^2u}{dt^2}=\dfrac{mu}{(x^2+u^2)^{\frac32}}=f$, since $\sigma=u$. But $v=Ft$, $v_1=ft$.

$$\therefore\ v_1=\frac{muv}{nx}.\quad \text{Also } \frac{a-x}{v}=\frac{b-u}{v_1}=\frac{(b-u)nx}{muv}.$$

$$\therefore\ u=\frac{bnx}{am-mx+nx}\ ..(1).\quad \therefore\ \frac{d^2x}{dt^2}=\frac{n(am-mx+nx)^3}{x^2[a^2n^2+(am-mx+nx)^2]^{\frac32}}\ ..(2).$$

From (1), $u=0$ when $x=0$. Therefore, the particles both arrive at O at the same time. Hence C moves over distance a, and D moves over distance b before they meet. The time is found by integrating (if possible) (2) twice.

$$\text{If } m=n,\ \frac{d^2x}{dt^2}=\frac{a^3m}{x^2\sqrt{[(a^2+b^2)^3]}}.$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2=\frac{2a^3m}{(a^2+b^2)^{\frac32}}\left(\frac1x-\frac1a\right)=\frac{2a^2m}{(a^2+b^2)^{\frac32}}\cdot\frac{a-x}{x}.$$

$$\therefore\ t=\frac{(a^2+b^2)^{\frac34}}{a\sqrt{2m}}\int_0^a\sqrt{\frac{x}{a-x}}\,dx=\frac{\pi(a^2+b^2)^{\frac34}}{2\sqrt{(2m)}}.$$

If $a=b$ and m=unity, $t=\pi(\frac12a^2)^{\frac34}$.

AVERAGE AND PROBABILITY.

92. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A circular field, radius r, is divided into four *equal* parts, by concentric circles and three concentric rings. From the center of this field are fired *at random*, and with such a velocity as not to produce a range greater than the radius of the field, $m=1000$ projectiles of the *same* kind. How many projectiles should have fallen into each one of these four equal parts of the field?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The range$=(v^2/g)\sin2\theta$, where θ=angle of elevation. Greatest range $=v^2/g=r$.

$\therefore v^2=gr$. $\therefore (v^2/g)\sin 2\theta = r\sin 2\theta$.

The radii of the three concentric circles are $\frac{1}{2}r\sqrt{3}$, $\frac{1}{2}r\sqrt{2}$, $\frac{1}{2}r$, respectively.

$\therefore r\sin 2\theta=\frac{1}{2}rn$, suppose. Solving, we get $\sin\theta=\frac{1}{2}\sqrt{[2\pm\sqrt{(4-n^2)}]}$.

When $n=\sqrt{3}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm1)}$. $\therefore \theta=\frac{1}{3}\pi$ or $\frac{1}{6}\pi$.

When $n=\sqrt{2}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{2})}$. $\therefore \theta=3\pi/8$ or $\frac{1}{8}\pi$.

When $n=1$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{3})}$. $\therefore \theta=5\pi/12$ or $\frac{1}{12}\pi$.

Chance that all fall into outer ring$=p$.

$$\therefore p=\frac{r\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}.\quad p_1=\frac{r\int_{\frac{1}{8}\pi}^{3\pi/8}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}\sqrt{2},$$

the chance that all will fall in the two outer rings.

$\therefore P=p_1-p=\frac{1}{2}(\sqrt{2}-1)$=the chance that all will fall in the second ring from without.

$$p_2=\frac{r\int_{\frac{1}{12}\pi}^{5\pi/12}\sin 2\theta d\theta}{r\int_0^{\frac{1}{2}\pi}\sin 2\theta d\theta}=\tfrac{1}{2}\sqrt{3},$$ the chance that all will fall in three outer rings.

$\therefore P_1=p_2-p_1=\frac{1}{2}(\sqrt{3}-\sqrt{2})$=chance that all fall in third ring from without.

$\therefore P_2=1-p_2=\frac{1}{2}(2-\sqrt{3})$=chance that all fall in small circle around the center.

$\therefore$ Number to fall in each space is proportional to these chances, or as $1:(\sqrt{2}-1):(\sqrt{3}-\sqrt{2}):(2-\sqrt{3})$.

$\therefore$ Number in outer ring$=\frac{1}{2}m=500$.

Number in second ring$=m(\sqrt{2}-1)/2=500(\sqrt{2}-1)=207.1065$.

Number in third ring$=m(\sqrt{3}-\sqrt{2})/2=500(\sqrt{3}-\sqrt{2})=158.9185$.

Number in inside circle$=m(2-\sqrt{3})/2=500(2-\sqrt{3})=133.9750$.

93. Proposed by **LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.**

In Problem 75, required the average area of the circle inscribed in the triangle.

Solution by **G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

Let r=radius of inscribed circle.

Then $\frac{1}{2}pr=\text{area}=\frac{1}{2}r[x+y+\sqrt{(x^2+y^2)}]=\frac{1}{2}xy$.

But $x=\dfrac{p^2-2py}{2(p-y)}$. $\therefore \frac{1}{2}pr=\dfrac{p(p-2y)y}{4(p-y)}$.

$$\therefore r=\frac{(p-2y)y}{2(p-y)},\quad \pi r^2=\frac{\pi y^2(p-2y)^2}{4(p-y)^2}.$$

The limits of y are 0 and $\dfrac{p}{2+\sqrt{2}}=y'$.

$$\therefore \triangle=\frac{\frac{1}{4}\pi\int_0^{y'}\frac{y^2(p-2y)^2}{(p-y)^2}dy}{\int_0^{y'}dy}=\frac{\pi(2+\sqrt{2})}{4p}\int_0^{y'}\left(4y^2+4py+5p^2+\frac{p^4}{(p-y)^2}-\frac{6p^3}{p-y}\right)dy$$

$$=\frac{\pi p^2}{12}[27-4\sqrt{2}-9(2+\sqrt{2})\log 2].$$

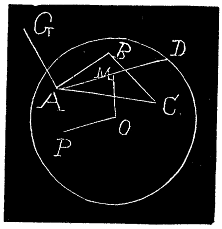
In this solution, as in solution of problem 75, I used the limits of y, 0 and $p/(2+\sqrt{2})$. These limits give all possible variations of size of area. Any other areas are mere repetitions of those included in the above and such a repetition or doubling of areas I believe to be inadmissible.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

Solution by the PROPOSER.

Let AD be the diameter of the section of the sphere made by the plane through the three random points A, B, C; M its center; O the center of the sphere; OP a line such that AB is parallel to the plane MOP; p=the chance.



Let $AO=r$, $\angle AOM=\theta$, $\angle GAC=\varphi$, $\angle GAB=\psi$, $\angle MOP=\lambda$, the angle MOP makes with some fixed plane through $OP=\rho$.

An element of the sphere at A is $4\pi r^2\sin\theta d\theta$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $\frac{1}{2}\pi$ and π; of ψ, $\pi-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π.

The three points can be taken $64\pi^3r^6$ ways on the surface of the sphere. Hence

$$p=\frac{1}{64\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{2\pi}\int_0^{\pi}\int_0^{2\pi}4\pi r^2\sin\theta d\theta . 4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)\sin\lambda d\theta d\varphi d\psi d\lambda$$

$$=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)d\theta d\varphi d\psi$$

$$=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\sin^2\theta(4\sin^2\varphi-4\sin^4\varphi+\pi\sin\varphi\cos\varphi-2\varphi\sin\varphi\cos\varphi)d\theta d\varphi$$

$$=\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\sin^3\theta d\theta=\tfrac{1}{3}.$$

MISCELLANEOUS.

88. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Solve to infinity the series $5\cos\theta=\frac{7\cos3\theta}{3\,!}+\frac{9\cos5\theta}{5\,!}\ldots\ldots$

Solution by the PROPOSER.

Let the series be $C=\frac{5\cos\theta}{1}+\frac{7\cos3\theta}{3\,!}+\frac{9\cos5\theta}{5\,!}+\ldots.$

also, $S=\frac{5\sin\theta}{1}+\frac{7\sin3\theta}{3\,!}+\frac{9\sin5\theta}{5\,!}+\ldots.$

Then $C+Si=\frac{5(\cos\theta+i\sin\theta)}{1}+\frac{7(\cos3\theta+i\sin3\theta)}{3\,!}+\ldots.$

or using a familiar notation,

$$C+Si=\frac{5e^{i\theta}}{1}+\frac{7\,e^{3i\theta}}{3\,!}+\frac{9\,e^{5i\theta}}{5\,!}+\ldots.$$

which can be written thus:

$$=4\left(e^{i\theta}+\frac{e^{3i\theta}}{3\,!}+\frac{e^{5i\theta}}{5\,!}+\ldots.\right)+\left(e^{i\theta}+\frac{3\,e^{3i\theta}}{3\,!}+\frac{5\,e^{5i\theta}}{5\,!}+\ldots.\right)$$

$$=4\left(e^{i\theta}+\frac{e^{3i\theta}}{3\,!}+\frac{e^{5i\theta}}{5\,!}+\ldots.\right)+e^{i\theta}\left(1+\frac{e^{2i\theta}}{2\,!}+\frac{e^{4i\theta}}{4\,!}+\ldots.\right)$$

$$=2\left(e^{e^{i\theta}}-e^{-e^{i\theta}}\right)+\frac{e^{i\theta}}{2}\left(e^{e^{i\theta}}+e^{-e^{i\theta}}\right)$$

But $e^{i\theta}=\cos\theta+i\sin\theta$, so that we have

$$C+Si=2\left(e^{\cos\theta+i\sin\theta}-e^{-\cos\theta-i\sin\theta}\right)+\frac{e^{i\theta}}{2}\left(e^{\cos\theta+i\sin\theta}+e^{-\cos\theta-i\sin\theta}\right)$$

$$=2e^{\cos\theta}\,e^{i\sin\theta}-2e^{-\cos\theta}\,e^{-i\sin\theta}+\frac{e^{\cos\theta}}{2}.e^{i\theta+i\sin\theta}+\frac{e^{-\cos\theta}}{2}\,e^{i\theta-i\sin\theta}$$

$$=2e^{\cos\theta}[\cos(\sin\theta)+i\sin(\sin\theta)]-2e^{-\cos\theta}[\cos(\sin\theta)-i\sin(\sin\theta)]$$

$$+\frac{e^{\cos\theta}}{2}\Big[\cos(\theta+\sin\theta)+i\sin(\theta+\sin\theta)\Big]+\frac{e^{-\cos\theta}}{2}\Big[\cos(\theta-\sin\theta)+i\sin(\theta-\sin\theta)\Big].$$

Now equating the real parts we have,

$$C=2e^{\cos\theta}[\cos(\sin\theta)]-2e^{-\cos\theta}[\cos(\sin\theta)]+\frac{e^{\cos\theta}}{2}[\cos(\theta+\sin\theta)]+\frac{e^{-\cos\theta}}{2}[\cos(\theta+\sin\theta)]$$

which is the required sum.

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

132. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

Solve $2^x+3^y=4$; $5^x+6^y=7$.

133. **Proposed by HARRY S. VANDIVER, Bala, Montgomery County. Pa.**

A theory of Fermat. The sum of two integral fourth powers cannot be an integral square. [Cf. *Chrystal's Algebra*, Vol. II., page 535.]

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

161. **Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.**

A circle, radius r, is inscribed in a triangle ABC. In the angles A, B, and C are inscribed circles each touching two sides and the inscribed circle. There are six such circles. The first group of three have their centers between the incenters and the vertices, and the second group of three does not. Let r_a, r_b, r_c denote the radii of the first group. Then this well known relation holds: $r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}$. Let R_a, R_b, R_c denote the radii of the second group. Then this relation holds:

$$\frac{1}{r}=\frac{1}{\sqrt{(R_aR_b)}}+\frac{1}{\sqrt{(R_bR_c)}}+\frac{1}{\sqrt{(R_cR_a)}}.$$

Required proof.

162. **Proposed by J. D. PALMER, Providence, Ky.**

Given the distances from the vertices of a triangle, ABC, to the center of the incircle, to construct the triangle.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

126. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find the volume contained between the conical surface whose equation is $z=a-\sqrt{(x^2+y^2)}$, and the planes whose equations are $x=z$ and $x=0$ by the formula $\int\int\int dxdydz$. [*Todhunter's Integral Calculus.*]

127. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the moment of inertia of a parallelogram about an axis perpendicular to its plane and passing through the intersection of its diagonals.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

115. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A vessel in the shape of a parallelopiped, filled with water, has in its horizontal bottom a rectangular opening, whose dimensions are a and b, which is shut up by a slider. Supposing this slider to be opened with a uniform motion in the direction of a. To find the depth of the water in the vessel after the time T at the moment when the slider has passed through the space a, a denoting the horizontal section of the water in the vessel.

116. Proposed by C. L. CHILTON, Greensboro, Ala.

Given, the shaft ABC attached at one end by a pivot to the piston-rod of an engine (at A) and the other to the crank of a wheel CDE (at C). The shaft moves through the distance of two feet in one second from A to B and at the same time turns the crank from C to E. The force propelling the shaft along the constrained course from A to B is 5760 pounds. The mass of the rod and wheel and friction being not considered, what would be the kinetic energy of the machine? or the sum of the moment around O, the center of the wheel?

117. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

How much lower must *one end* of a heavy uniform chain, wound round the circumference of a perfectly rough vertical wheel, hang than *the other end*, when the chain is on the point of motion?

118. Proposed by M. E. ANDERSON, Minneapolis, Minn.

A closed steel cylinder of length L and diameter D is placed in a horizontal position. The cylinder is filled with water to a depth (a) from the lower side, the space above the water being filled with air at a pressure P_1.

What work will be done against this increasing pressure, and against gravity, by a pump forcing water into this tank until the pressure has increased to P_2? Suppose the level of the water in the tank at the beginning to be the same as that of the reservoir from which the water is pumped.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

AVERAGE AND PROBABILITY.

101. **Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.**

By direct calculation obtain the average distance between to points in the surface of a circle.

102. **Proposed by PROFESSOR CAVALLIN.**

A random straight line is determined by two points taken at random within a sphere; find the average velocity acquired by a particle in descending the line. [No. 6742, *Educational Times*. Unsolved.]

93. **Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.**

A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

102. **Proposed by CHARLES C. CROSS, Whaleyville, Va.**

Required the least multiple of 17 which when divided by 2, 3, 4, 5, ... 16, leaves, in each case, 1 as a remainder.

103. **Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.**

Solve, log sin x=sin log x.

104. **Proposed by HARRY S. VANDIVER, Bala, Pa.**

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. *Chrystal's Algebra*, Vol. II., page 535.]

*** Solutions of these problems should be sent to J, M. Colaw not later than May 10.

NOTES.

Gustav Fock is offering for sale the very valuable mathematical library of Dr. Bruno Christoffel.

Professor M. Cantor of Heidelberg, has been elected a correspondent of the St. Petersburg Academy of Science.

Professor Henry S. White, of Northwestern University, has received leave of absence and will remain abroad until October.

In Italy has just appeared a new mathematical journal issued at Città di Castello by the publisher, S. Lapi, to whom the annual subscription, 12 francs, should be sent. It is a monthly magazine called *Le Matematiche*, under the direction of Prof. C. Alasia with a board of collaborators among whom the English language is represented by G. B. Halsted of Austin, Texas, to whom communications may be sent, which will appear in Italian. On the Editorial Board may also be noted the Russian, Vasiliev, and one of the greatest of living mathematicians, Poincaré. The first number, February, 1901, contains the last thing ever written for publication by the illustrious Hermite, dated January, 1901, on the 14th of which month he died. This number honors THE AMERICAN MATHEMATICAL MONTHLY by reproducing from it in Italian an interesting note. A new and very suggestive department is introduced under the heading, "Subjects for Research."

BOOKS AND PERIODICALS.

Elements of Physics. By Henry A. Rowland, Ph. D., LL. D., Professor of Physics, and Director of the Physical Laboratory in Johns Hopkins University, and Joseph S. Ames, Ph. D., Professor of Physics and Sub-Director of the Physical Laboratory in Johns Hopkins University. 8vo, cloth, xiii+263 pages. Price, $1.00. New York and Chicago: The American Book Company.

This text-book is designed to meet the requirements of high schools and normal schools in the subject of Physics and it is based on the principle that the object of physics is two-fold, viz., (1) to train the student in the powers of observation and accurate description, and (2) to cultivate the habits of exact thought and statement. Great emphasis has been put on those points in the study of the subject which are necessary for the mental training of the student and which will make the more elaborate discussions of the subject simpler when the student comes to them. The reputation of its authors assures the highest authority of statement and great care and thought in its preparation.

B. F. F.

The Common Sense of Commercial Arithmetic. By George Hall, Principal of Petersburg Academy, Petersburg, Va. 8vo. Cloth, xii+187 pages. Price, 60 cents. New York: The Macmillan Co.

This book is designed to give pupils a definite idea of the principles of common sense underlying the subject of Arithmetic common to commercial life. Each subject is clearly and carefully treated, many examples solved in complete detail, and many original problems of practical occurence are proposed for solution. The author's method of presenting percentage is good and his ideas on the subject are sound. B. F. F.

Non-Euclidean Geometry. By Henry Parker Manning, Ph. D., Assistant Professor of Mathematics in Brown University. 8vo, cloth, 94 pages. Boston and Chicago: Ginn & Co.

This is the first attempt in America to present the Non-Euclidean Geometry in a form suitable for use in schools and colleges. The author lays no claim to originality as to subject matter, as much of the work came to the author through the translations of Dr. Halsted. We believe this little book will do much towards popularizing the subject and thus will bring it within the comprehension of teachers of Geometry. In our next issue Dr. Halsted will give an extended review of the work. B. F. F.

The Free Expansion of Gases. Memoirs by Gay-Lussac, Joule ,and Joule and Thompson. Translated and edited by J. S. Ames, Ph. D., Professor of Physics in Johns Hopkins University. 8vo. cloth, 106 pages. Price, $0.75. New York and Chicago : The American Book Co.

This is one of the Series of Memoirs on Physical Subjects, the publication of which was undertaken a few years ago by Harper & Bros. The Series comprises (1) Prismatic and Diffraction Spectra, (2) The Free Expansion of Gases, (3) The Röntgen Rays, (4) The Modern Theory of Solution, and (5) The Laws of Gases. These important memoirs should be in the hands of every teacher of Physics. B. F. F.

Theory of Thought and Knowledge. By Borden P. Bowne, Professor of Philosophy in Boston University. Large 8vo. cloth. xiii+389 pages. Price, $1.50. New York and Chicago: The American Book Co.

This work is written by one of America's ablest thinkers. Professor Bowne deals with the subtilest problems of philosophy in a masterly way. This book consists of two parts; the first part, covering 263 pages, deals with *Theory of Thought* and the second part with the *Theory of Knowledge.* This book is invaluable to the student of Philosophy.
B. F. F.

Le Mathematiche. Pure ed Applicate. Periodico mensile di matematiche pure ed applicate, superioai ed elementari, ad uso dell' isturzione media e superore diretto dal Prof. Cristoforo Alasia, con la Collaborazione dei piu illustri Scienziati Italiani e Stranieri.

This monthly journal has just been started, the February number being the first issue. Professor Cristoforo Alasia, the editor, has a large number of very able mathematicians, both native and foreign, to assist him, among whom is Dr. Halsted, who represents America. The MONTHLY wishes this journal the greatest success. B. F. F.

Mathemaitsch-naturwissenschaftliche Mitteilungen be gründet von Dr. O. Böklen, im auftrag des mathematisch-naturwissenschaftlichen Vereins in Württemberg herausgegeben von Dr. A. Schmidt, Dr. A. Haas, und Dr. E. Wölffing.

Inhalt, Januar, 1901. O Böklen, von Dr. E. Wölffing; Vereinsnachrichten; Die nicht-euclidesche Geometrie und die Trigonometrie auf den Flächen von konstantem Krümmungsmass, von Dr. K. Kommerell.

The Mathematical Messenger, Edited and Published by G. H. Harvill, A. M., Athens, Texas.

After a long period of suspension, this journal again makes its appearance. It is published bi-monthly and is devoted chiefly to solutions of problems. The price of the journal is two dollars per year. B. F. F.

Periodico di Mathematica for January and February contains an extensive discussion of the "last theorem of Fermat."

The Mathematical Gazette. Edited by W. J. Greenstreet, M. A. with the coöperation of F. S. McCauley, M. A., D. Sc., and others.

This magazine is published bi-monthly and contains articles, correspondence notes, reviews of books, and problems and solutions. B. F. F.

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NOTE ON POLES AND POLARS.

By GEORGE R. DEAN, C. E., B. Sc., Professor of Mathematics, Missouri School of Mines and Metallurgy, Rolla, Mo.

While the matter here presented is not new to the mathematician, the method appears, from a pedagogical point of view, to have some advantages.

Through the point $O\equiv(x', y')$ a straight line is drawn cutting a conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at A and B.

Let C be a point in AB such that $OC=\dfrac{2OA\times OB}{OA+OB}$.

We propose to find the locus of C, when the straight line turns about the point O.

Transferring the origin to (x', y') the new equation is

$$ax^2+2hxy+by^2+2(ax'+hy'+g)+2(hx'+by'+f)+$$
$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0.$$

For the sake of brevity we write this

$$ax^2+2hx'y'+by'^2+2g'x'+2f'y'+c'=0,$$

where $g'=ax'+hy'+g$, $f'=hx'+by'+f$, $c'=ax'^2+2hx'y'+$ etc.

Putting $x=r\cos\theta$, $y=r\sin\theta$, and arranging according to powers of r,

$$(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)r^2+2(g'\cos\theta+f'\sin\theta)+c'=0.$$

Then $OC = \dfrac{2 \text{ product of roots}}{\text{sum of roots}} = \dfrac{-c'}{g'\cos\theta + f'\sin\theta}$.

Putting $OC=\rho$, we have $g'\rho\cos\theta + f'\rho\sin\theta + c' = 0$.

But $\rho\cos\theta = x$, $\rho\sin\theta = y$; hence $g'x + f'y + c' = 0$.

This straight line is the polar of the point (0, 0) with respect to the conic $ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$.

Transferring to the old origin the equation of the polar becomes

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')+ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0,$$

which may be written

$$axx'+h(xy'+x'y)+bgy'+g(x+x')+f(y+y')+c=0.$$

This is the equation of the polar of $x'y'$ with respect to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If the point C be fixed then will O describe the polar of C; for, we have on putting $OC=r$, $CA=a$, $CB=b$,

$$r=\frac{2(r+a)(r+b)}{2r+a+b}.$$

Clearing and solving for r, we find $-r=\dfrac{2ab}{a+b}$, *i. e.* $CO=\dfrac{2CA \times CB}{CA+CB}$.

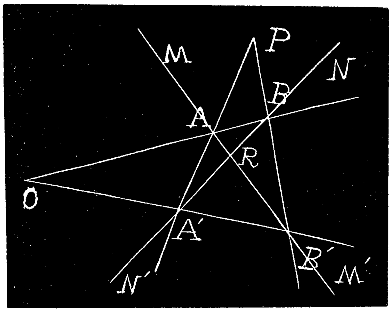
Hence: *If the point P lies on the polar of the point Q, then will Q lie on the polar of P.*

If the line OAB is drawn tangent to the curve the points A and B coincide with C. Hence:

The polar of a point passes through the points of contact of the tangents drawn through the point.

If O be taken on the curve, C will coincide with O, and the polar is the tangent at O.

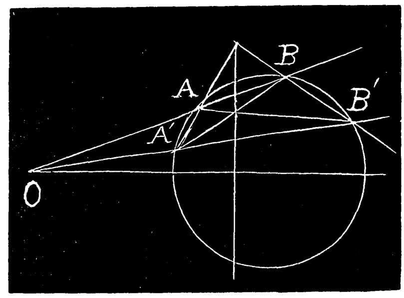
The polar of a point with respect to a pair of straight lines passes through the intersection of the lines; for, on drawing a line through O and the point of intersection, A and B coincide with C.



To draw the polar of a point with respect to a pair of straight lines: Through O draw two secants cutting the lines at A and B, A' and B', respectively. Join AA', BB' and produce till they meet at P. Join P with common point of given lines R. PR is the polar. For, the polar of AA' and BB' passes through P and cuts AB and $A'B'$ at the same points as the polar with respect to NN' and MM'. The line OR is the polar of P with respect to either of the two pairs of

lines. OP is the polar of R with respect to the same lines ; in other words the triangle OPR is self-conjugate.

To draw the polar of a point with respect to a conic: Draw any two secants through O, join AB', $A'B$, AA', BB'. The polar with respect to the conic is the same as the polar with respect to AB' and $A'B$, or with respect to AA' BB'.



Construction of Pole and Polar when center is given: The tangents at points (x_1, y_1) (x_2, y_2) of the curve are

$$axx_1+h(xy_1+x_1y)+byy_1+$$
$$g(x+x_1)+f(y+y_1)+c=0,$$

$$axx_2+h(xy_2+x_2y)+byy_2+g(x+x_2)+f(y+y_2)+c=0.$$

Subtraction gives

$$ax(x_1-x_2)+hx(y_1-y_2)+hy(x_1-x_2)+by(y_1-y_2)+g(x_1-x_2)+f(y_1-y_2)=0,$$

$$\text{or } (ax+hy+g)(x_1-x_2)+(hx+by+f)(y_1-y_2)=0,$$

the equation of a line through the intersection of the tangents. This line passes through the center, since the coördinates of the center cause $ax+hy+g$, $hx+by+f$, to vanish.

Putting $y_1-y_2=m(x_1-x_2)$, we may write this equation

$$(a+hm)x+(h+bm)y+(g+fm)=0.$$

This is the equation of the diameter which bisects the chord.

This may also be shown by putting $\dfrac{x_1+x_2}{2}$ for x, $\dfrac{y_1+y_2}{2}$ for y, the equation being satisfied thereby.

Finally: *The pole of a given line lies on the diameter which bisects the chord intercepted on that line by the conic.*

To find the pole of a given line : Draw the diameter conjugate to the line, join its extremities with any point of the line produced to cut the conic. Through the points of intersection draw a line to cut the diameter.

The construction in case of the parabola is easily effected by remembering that the center is at infinity.

A CLASS-BOOK OF NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

America has taken a step in advance of all the world. She now has a text-book, a manual for class use, in non-Euclidean geometry. It is a pleasure to point out the close connection of THE AMERICAN MATHEMATICAL MONTHLY with this note-worthy achievement.

In this book, *Non-Euclidean Geometry*, by Henry Parker Manning, Ph. D., Assistant Professor of Pure Mathematics in Brown University, Ginn & Co., 1901, Pp. v+95, the method of treatment that has been taken as the basis of the first chapter, and which consequently underlies the other three chapters of the book, is that of Saccheri, drawn directly and solely from the pages of THE AMERICAN MATHEMATICAL MONTHLY where the first translation of Saccheri began to be published in June, 1894.

THE MONTHLY, Vol. I, p. 188, gives Saccheri's first proposition literally: "If two equal straights [sects] AC, BD, make with the straight AB angles equal toward the same parts: I say the angles at the join CD will be mutually equal." On page 189 is "Proposition II. The quadrilateral $ABCD$ remaining the same, the sides AB, CD are bisected in points M, and H. I say the angles at the join MH will be on both sides right."

Professor Manning paraphrases these two together on page 5: "If two equal lines in a plane are erected perpendicular to a given line, the line joining their extremities makes equal angles with them and is bisected at right angles by a third perpendicular erected midway between them."

In some respects we prefer the phraseology of Saccheri to that of Manning. Like the French, neither possesses a word for a piece of a straight line, German strecke, English *sect*. There is a kind of unbounded entity such that one and only one is on two points. This may be called *the straight*. It appears as an element in projective geometry. But it is of the essence of metric geometry that two points shall also completely determine an entity bounded by them, *the sect*, with which the idea of precise individual magnitude or quantity may be connected by setting up a conventional system of measurement. Distance is the result of the comparison of two sects. The distance between two points is the length of their sect in terms of a standard sect, say the centimeter.

Both the accepted popular and the accepted mathematical definitions of distance make it always a number, as, *e. g.* Wentworth, 1899, page 8, §50. "Def. The *distance* between two points is *the length* of the straight line [sect] that joins them;" and again the Cayley-Klein definition: "The distance between two points is equal to a constant times the logarithm of the cross-ratio in which the line joining the two points is divided by the fundamental quadric." Saccheri calls the two equal sects of his first proposition *straights*. Manning goes farther off, calling them *lines*.

B. A. W. Russell in "An Essay on the Foundations of Geometry" uses

the word 'distance' as a confounding and confusing designation for a sect itself and also for the numerical measures of that sect, whether by superposition, ordinary ratio, indeterminate as depending on the choice of a unit, or by projective metrics, indeterminate as depending on the fixing of the three points to be taken as constant in the varying cross ratios, these cross ratios themselves to be defined as numbers by the method of von Staudt, without presupposing ordinary measurement.

The confusion which may thus be introdued just from lack of a word is powerfully shown by the illustrious Poincaré in a response to Mr. Russell in the *Revue de Métaphysique et de Morale*, 1900, pp. 73—86, entitled "Sur les principes de la Géométrie." The following four sentences are curious in showing his results and at the same time showing the lack in French which causes a borrowing from German: "Comme le fait remarquer M. Halsted dans une brochure récente (*Science*, N. S. Vol. X, No. 251), M. Russell a eu tort d'employer indifférement le mot distance pour désigner ce que les Allemands appelent 'Strecke,' et en même temps la mesure de cette 'Strecke.' Le nom de distance ne convient qu' à la mesure de cette 'Strecke,' et cette mesure ne peut être définie que par une convention.

Si M. Russell n' avait pas, comme le lui reproche M. Halsted (vide supra), employé le mot distance dans deux sens différents, il n' y aurait plus d' apparence de cercle vicieux. Où serait cette apparence si l' on avait écrit: La distance est le résultat de la comparison de deux 'Strecken'."

Again, in both propositions Saccheri speaks of the *join* of two points. Manning paraphrases it as "the line joining" the two points. In a note to Euclid I. 5, Todhunter says of the phrase "Join FC"; "Custom seems to allow this singular expression as an abbreviation for 'draw the straight line FC', or for 'join F to C by the straight line FC.'" In Saccheri the join AB means the sect terminated by A and B. In projective geometry the join AB means the unbounded straight on A and B.

Under the heading *Definitions*, Saccheri says: "Since (from our first) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we will call base), makes equal angles with these perpendiculars, three hypotheses are to be distinguished according to the species of these angles. And the first, indeed, I will call hypothesis of right angle; the second, however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle." This Manning paraphrases as follows, under the heading *The Three Hypotheses*: "The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures. We shall distinguish the three cases by speaking of them as the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle, respectively."

Saccheri's Prop. III. is: "If two equal straights, AC, BD, stand perpendicular to any straight, AB: I say the join CD will be equal, or less, or greater than AB, according as the angles at CD are right, or obtuse, or acute."

This Manning paraphrases as follows : "The line joining the extremities of two equal perpendiculars is, at least for any restricted portion of the plane, equal to, greater than, or less than the line joining their feet in the three hypotheses respectively."

In the same way is paraphrased Saccheri's Prop. IV., the converse of III.

Saccheri's Corollary about quadrilaterals with three right angles is given, page 12.

Saccheri's Prop. V. is : "The hypothesis of right angle, if even in a single case it is true, always in every case alone is true." In giving this, Manning writes : "If the hypothesis of *a* right angle," &c, evidently a slip for *the* right angle. Of course the Latin has no article.

Prop. VI. and Prop. VII. are combined, p. 13. Prop. IX. is reproduced on p 14. Prop. X. is given on p. 9.

In Prop. XI. Saccheri with the hypothesis of right angle demonstrates the celebrated Postulatum of Euclid, thus showing that his hypothesis of right angle is the ordinary Euclidean geometry. Manning does not reproduce this demonstration but says, p. 27 : "The three hypotheses give rise to three systems of Geometry, which are called the Parabolic, the Hyperbolic, and the Elliptic Geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann." It should be noted that Manning's book gives only the simple elliptic, or single elliptic, or Clifford-Klein Geometry. It never even mentions the double elliptic or Spherical or Riemannian Geometry, which Killing maintains was the only form which ever came before Riemann's mind.

Manning's Chapter II., The Hyperbolic Geometry, seems taken bodily from Halsted's translation of Lobachevski's "Geometrical Researches on the Theory of Parallels." Though Halsted's translation of Bolyai is specifically mentioned on page 94, yet Manning shows no signs of having read it, and thus his book is confined within the bounds of propædeutics.

The most extraordinary two dozen pages in the history of thought is "The Science Absolute of Space," by Bolgai János. This is the most perfect case of genius. Take as example his §34 : Through a given point to draw a parallel to a given straight. So simple; and yet neither Lobachevski nor anyone else ever reached it. It seems supernatural, uncanny. it makes one's hair stand on end. Perhaps he was, as he called himself, the phœnix of Euclid.

Manning's Chapter III., The Elliptic Geometry, pp. 62-8, is very brief, but what there is of it is good. The final Chapter IV., Analytic Non-Euclidean Geometry, is devoted to putting into coördinate and equational notation the new matters reached synthetically in the preceding chapters. The book ends with a Historical Note, pp. 91-5. This, in the main sound, may be in parts misleading. Thus it says, p. 91 ; "Legendre proved that the sum of the angles of a triangle can never exceed two right angles, and that if there is a single triangle in which this sum is equal to two right angles, the same is true of all triangles. This was, of course, on the supposition that a line is of infinite length." Now this beautiful theorem, if a single, then all triangles, Manning has in his own

book, p. 12, under the form "If the hypothesis of [a] right angle is true in a single case, it holds true in every case." proved by Saccheri without any supposition on the length of the line and a century before Legendre.

Again, not even the name of Schweikart is mentioned, though as I have shown in THE AMERICAN MATHEMATICAL MONTHLY, Vol. VII., pp. 247-252, and in "Science," Vol. XII., pp. 842-846, Schweikart may be considered the first to publish a genuine conscious treatise on Non-Euclidean Geometry [which I there give for the first time in English]. This fixes the date of the first conscious creation and naming of the Non-Euclidean Geometry as between 1812 and 1816.

We may perhaps timidly hope that the great Jesuit, Saccheri, had some suspicion of what he had really done. But meanwhile it seems almost certain that he really believed that his beautiful Non-Euclidean Geometry was all a reductio-ad-absurdum, and that he had really justified the title of his book, "Euclid Vindicated from Every Fleck," by proving Non-Euclidean Geometry untenable. On the other hand we of the new school, followers of Schweikart and Bolgai János, believe that it is our Non-Euclidean Geometry itself which finally vindicates Euclid from every fleck, and justifies the weighty tribute of Professor Alfred Baker: "Of the perfection of Euclid (B. C. 290) as a scientific treatise, of the marvel that such a work could have been produced two thousand years ago, I shall not here delay to speak. I content myself with making the claim that, as a historical study, Euclid is, perhaps, the most valuable of those that are taken up in our educational institutions."

Austin, Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

115. Proposed by **ALOIS F. KOVARIK**, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Find the conditions of the coefficients of a general biquadratic equation so that it may be solved by quadratics.

I. Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the **PROPOSER**.

To solve as a pure quadratic, without using the cubic, we might proceed as follows:

Let $x^4+ax^3+bx^2+cx+d=0$ be the general biquadratic.

(1) Then $(x^2+\frac{1}{2}ax)^2+bx^2-\frac{1}{4}a^2x^2+cx+d=0$.

Now if $c=\frac{1}{2}ab-\frac{1}{8}a^3$, we get $(x^2+\frac{1}{2}ax)^2+(b-\frac{1}{4}a^2)(x^2+\frac{1}{2}ax)+d=0$.

This equation can be solved by quadratics.

(2) Let $d=c^2/a^2$. Then $x^4+ax^3+bx^2+cx+c^2/a^2=0$.

$\therefore (x^2+\frac{p}{2}x+\frac{c}{a})^2=(\frac{p^2}{4}+\frac{2c}{a}-b)x^2$.

This is easily solved by quadratics.

This gives two conditions: 1st, $c=\frac{1}{2}ab-\frac{1}{8}a^3$; $d=c^2/a^2$.

II. Solution by HARRY S. VANDIVER, Bala, Pa.

The problem, I take it, is equivalent to the following:

What is the condition that the general biquadratic equation has a root which may be expressed in quadratic surds?

(A) Suppose the equation to be reduced to the form

$$x^4+qx^2+rx+s=0\ldots.(1),$$

where q, r, and s are rational; and assume that this is equivalent to

$$(x^2-kx+m)(x^2+kx+l)=0\ldots.(2);$$

then, following Descartes' method, we find as the cubic resolvent in k^2,

$$k^6+2qk^4+(q^2-4s)k^2-r^2=0\ldots.(3).$$

(*Hall and Knight's Higher Algebra*, page 485).

Now since x is to be found in terms of k, l, and m from (2), and, if it is to be in the form of a quadratic surd, then (3) must have a root in the form of a quadratic surd. Therefore, the roots of (3) must be of the forms $\pm\sqrt{b}$, and $\pm\sqrt{(c\pm\sqrt{d})}$, where b, c, and d are rational. It is sufficient to consider one value of k; put $k=\sqrt{b}$, and substituting in (2), we obtain results of the form

$$x=\begin{cases}\sqrt{m}+\sqrt{(g+n\sqrt{m})}\ldots.(4),\\ \sqrt{m}-\sqrt{(g+n\sqrt{m})}\ldots.(5),\\ -\sqrt{m}+\sqrt{(g-n\sqrt{m})}\ldots.(6),\\ -\sqrt{m}-\sqrt{(g-n\sqrt{m})}\ldots.(7),\end{cases}$$

where m, g, and n are rational.

These values may be taken as the general forms of quadratic surd roots of biquadratics. Hence the *condition that the general biquadratic has roots of the form* (4), (5), (6), and (7) *is that* (3) *has a root in the form* $\sqrt{b}$, *where* b *is rational.*

(B) *Special forms of biquadratics. Relations existing between the coefficients.*

Let the general biquadratic be

$$x^4+px^3+qx^2+rx+s=0\ldots.(9),$$

and assume that it can be thrown into the form

$$(ax^2+bx)^2+c(ax^2+bx)+d=0\ldots\ (10);$$

then, expanding, and removing the term involving x^3 (by the proper linear transformation) the coefficient of x also vanishes, and from this property we find that the relation between the coefficients of (9) is :

$$p^3-4pq+8r=0,$$

and in (10), the sum of two of the roots is equal to the sum of the other two.

(Cf. Burnside and Panton's *Theory of Equations*, page 41).

2nd Case. Assume that (9) can be put in the form

$$(mx+\frac{n}{x})^2+g(mx+\frac{n}{x})+f=0\ldots.(11).$$

Multiplying out, and comparing with (9), we find that the relation between the coefficients is $r^2=p^2s$, *and the roots of* (11) *are in geometrical progression.*

Also solved by *CHARLES C. CROSS*, *B. L. REMICK*, *J. SCHEFFER*, *H. C. WHITAKER*, and *J. W. YOUNG.*

116. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equations :

$$w(xy+yz+xz)=a\ ;\ x(wy+wz+yz)=b\ ;$$

$$y(wx+wz+xz)=c\ ;\ z(wx+wy+xy)=d.$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. M. HOWIE, The Nebraska State Normal School, Peru, Neb.; B. L. REMICK, Bradley Polytechnic Institute, Peoria, Ill.; NOAH ADAIR, A. M., Tyler College, Tyler, Tex.; G. I. HOPKINS, High School, Manchester, N.H.; FREMONT CRANE, B. S., C. E., Stockett, Mont.; H. C. WHITAKER, Ph.D., Manual Training School, Philadelphia, Pa., C. C. CROSS, Meredithville, Va.; L. B. FILLMAN, Chester, Pa.; O. S. WESTCOTT, A. M., Sc. D., North Division High School, Chicago, Ill.; and W. W. LANDIS, A. M., Dickinson College, Carlisle, Pa.

(1)+(2)+(3)+(4) gives $wxy+wxz+wyz+xyz=\frac{1}{3}(a+b+c+d)\ldots.(5)$.

(5)−(1) gives $xyz=\frac{1}{3}(b+c+d-2a)\ldots.(6)$.

(5)−(2) gives $wyz=\frac{1}{3}(a+c+d-2b)\ldots.(7)$.

(5)−(3) gives $wxz=\frac{1}{3}(a+b+d-2c)\ldots.(8)$.

(5)−(4) gives $wxy=\frac{1}{3}(a+b+c-2d)\ldots.(9)$.

(6)÷(7) gives $x/w=(b+c+d-2a)/(a+c+d-2b)$.

(6)÷(8) gives $y/w=(b+c+d-2a)/(a+b+d-2c)$.

(6)÷(9) gives $z/w=(b+c+d-2a)/(a+b+c-2d)$.

These values of x, y, and z in (1) give

$$w=\sqrt[3]{\frac{(a+b+c-2d)(a+b+d-2c)(a+c+d-2b)}{3(b+c+d-2a)^2}}=\sqrt[3]{\frac{BCD}{3A^2}},\text{ suppose.}$$

Similarly, $x=\sqrt[3]{(ACD/3B^2)}$, $y=\sqrt[3]{(ABD/3C^2)}$, $z=\sqrt[3]{(ABC/3D^2)}$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; M. A. GRUBER, A. M., War Department, Washington, D. C.; J. H. DRUMMOND, LL. D., Portland, Me.; ELMER SCHUYLER, B. Sc., Boys' High School, Reading, Pa.; COOPER D. SCHMITT, M. A., University of Tennessee, Knoxville, Tenn.; and H. S. VANDIVER, Bala, Pa.

Putting $xyzw=P$, the given equations change into the following:

$$P/z+P/y+P/x=a,\ P/z+P/y+P/w=b,$$

$$P/z+P/x+P/w=c,\ P/y+P/x+P/w=d.$$

Adding, we get $P/x+P/y+P/z+P/w=\dfrac{a+b+c+d}{3}$.

Putting $\dfrac{a+b+c+d}{3}=s$, and subtracting each of the above equations, gives

$$P/w=s-a,\ P/x=s-b,\ P/y=s-c,\ P/z=s-d.$$

Multiplying, $P^3=(s-a)(s-b)(s-c)(s-d)$.

$$\therefore\ x=\sqrt[3]{\frac{(s-a)(s-c)(s-d)}{(s-b)^2}},\ y=\sqrt[3]{\frac{(s-a)(s-b)(s-d)}{(s-c)^2}},$$

$$z=\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{(s-d)^2}},\ w=\sqrt[3]{\frac{(s-b)(s-c)(s-d)}{(s-a)^2}}.$$

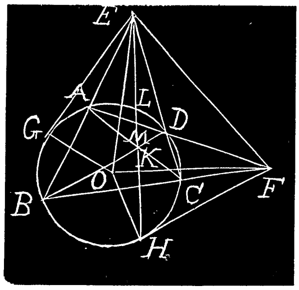
GEOMETRY.

146. Proposed by H. R. HIGLEY, M. Sc., Professor of Mathematics, Normal School, East Stroudsburg, Pa.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the line joining the points of concurrence=the sum of the squares on the two tangents from these points. Ex. 24, page 219, Mackay's *Elements of Euclid*.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $ABCD$ be the inscribed quadrilateral; E, F the intersection of the opposite sides produced; O the center of the circle; M the intersection of the diagonals; EG tangent from E; FH, FL tangents from F.



Draw EF, EM and let EM cut FO in K. It has been shown that EM is the polar of F.

$\therefore$ EM passes through L, H and is perpendicular to OF.

$\therefore$ $EF^2=FK^2+FK^2$.

But $EK^2=EO^2-OK^2$, $FK^2=(FO-OK)^2$.

$\therefore$ $FK^2=FO^2-2FO.OK+OK^2$.

But $2FO.OK=2OH^2=2r^2$. $\therefore$ $EK^2+FK^2=EO^2+FO^2-2r^2$.

$EG^2=EO^2-r^2$, $FH^2=FO^2-r^2$.

$\therefore$ $EG^2+FH^2=EO^2+FO^2-2r^2=EK^2+FK^2=EF^2$.

147. **Proposed by R. A. WELLS, Professor of Mathematics, Franklin College, Athens, Ohio.**

Find the locus in space of the point which is equally illuminated by each of two unequal lights whose intensities are a and b $(a>b)$, placed at a distance c from each other.

Solution by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let (x, y, z) be the rectangular coördinates of any one of the illuminated points, the mid-point of the line joining the two lights taken along the x-axis as origin; then the coördinates of the two lights may be given as $(l, 0, 0)$, $(-l, 0, 0)$, and by theory

$$\frac{a}{(x-l)^2+y^2+z^2}=\frac{b}{(x+l)^2+y^2+z^2}\ldots.(1),$$

or $x^2+y^2+z^2+2l\frac{a+b}{a-b}x+l^2=0\ldots.(2)$, a sphere.

Also solved by *G. B. M. ZERR*, and *J. SCHEFFER.*

148. **Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics in Syracuse University, Syracuse, N. Y.**

The condition that two triangles, abc, xyz, are similar is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix}=0,$$

and the condition that the triangle abc is equilateral is

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=0.$$

(Used in solving 130.)

I. **Solution by the PROPOSER.**

First. *The given conditions are necessary*, for if two triangles abc, xyz, be similar, then $\frac{a-b}{x-y}=\frac{b-c}{y-z}=\frac{c-a}{z-x}=r\ldots.(1)$, for this expresses not only that the homologous sides are proportional, but also that the homologous angles are equal. (1) is equivalent to the equations

$$\begin{aligned} a-b&=r(x-y) \\ b-c&=r(y-z) \quad \ldots(2). \\ c-a&=r(z-x) \end{aligned}$$

Multiplying the equations (2) by x, y, and z, respectively, and adding, we get

$$\Sigma(a-b)z=r\Sigma(x-y)z=0\ldots.(3),$$

or $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix}=0\ldots.(4)$, which is therefore necessary at least, if the triangles are similar.

If the triangle abc is equilateral, it is also equiangular, and

$$\frac{a-b}{b-c}=\frac{b-c}{c-a}=\frac{c-a}{a-b}=r\ldots.(5), \text{ or}$$

$$\begin{aligned} a-b&=r(b-c)\\ b-c&=r(c-a)\\ c-a&=r(a-b)\end{aligned}\quad\ldots.(6).$$

Multiplying these equations by a, b, c, respectively, and adding, we have

$$\Sigma a^2-\Sigma ab=r\Sigma(b-c)a=0\ldots.(7), \text{ or}$$

$$\begin{vmatrix} a & b & 1\\ b & c & 1\\ c & a & 1\end{vmatrix}=0\ldots.(8),$$

and this condition is also at least necessary.

Second. *These conditions are also sufficient.* For, with

$$\frac{a-b}{x-y}=r_1,\quad \frac{b-c}{y-z}=r_2,\quad \frac{c-a}{z-x}=r_3\ldots.(9),$$

or what is the same thing, with

$$\begin{aligned} a-b&=r_1(x-y)\\ b-c&=r_2(y-z)\\ c-a&=r_3(z-x)\end{aligned}\quad\ldots.(10)$$

$$\text{form }\begin{vmatrix} 1 & 1 & 1\\ a & b & c\\ x & y & z\end{vmatrix}=(a-b)z+(b-c)x+(c-a)y$$

$$=r_1z(x-y)+r_2x(y-z)+r_3y(z-x)\ldots.(11),$$

and put this equal to zero. We are then to prove in fact that this obliges r_1, r_2, r_3 to be equal to one another. We have

$$r_1z(x-y)+r_2x(y-z)+r_3y(z-x)=0\ldots.(12).$$

$$r_1(x-y)+r_2(y-z)+r_3(z-x)=0\ldots.(13).$$

The latter equation comes by adding equations (10) together. By solving (12) and (13),

$$\frac{r_1}{r_3}=\frac{\begin{vmatrix} x(y-z) & y(z-x) \\ (y-z) & (z-x) \end{vmatrix}}{\begin{vmatrix} z(x-y) & x(y-z) \\ (x-y) & (y-z) \end{vmatrix}}=\frac{(z-x)\begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix}}{(x-y)\begin{vmatrix} z & x \\ 1 & 1 \end{vmatrix}}=1.$$

$$\frac{r_2}{r_3}=\frac{\begin{vmatrix} y(z-x) & z(x-y) \\ (z-x) & (x-y) \end{vmatrix}}{\begin{vmatrix} z(x-y) & x(y-z) \\ (x-y) & (y-z) \end{vmatrix}}=\frac{(z-x)\begin{vmatrix} y & z \\ 1 & 1 \end{vmatrix}}{(y-z)\begin{vmatrix} z & x \\ 1 & 1 \end{vmatrix}}=1. \quad \ldots.(14).$$

or $r_1=r_3$, $r_2=r_3$, and therefore $r_1=r_2=r_3$. . .(15), whence the homologous sides of the triangles abc, xyz are proportional, and their homologous angles are equal, and the triangles are in fact similar. In the other case form

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=r_1a(b-c)+r_2b(-a)+r_3c(a-b)$$

$$\text{where } r_1=\frac{a-b}{b-c},\quad r_2=\frac{b-c}{c-a},\quad r_3=\frac{c-a}{a-b}\ \ldots.(16),$$

and put this equal to zero. We have

$$r_1a(b-c)+r_2b(c-a)+r_3c(a-b)=0.$$
$$r_1(b-c)+r_2(c-a)+r_3(a-b)=0. \quad \ldots.(17).$$
$$r_1r_2r_3=1.$$

The last two of equations (17) come from (16). As before, by solving the first two equations of (17) for r_1/r_3, r_2/r_3, we get $r_1=r_2=r_3=r$ say; by the last of equation (17) $r^3=1$, therefore

$$r=1$$
$$r=\cos 60^\circ+i\sin 60^\circ \quad \ldots.(18),$$
$$r=\cos 120^\circ+i\sin 120^\circ$$

and the only solution applying to a geometrical triangle in the plane is the second, which in fact determines an equilateral triangle, since modulus $r=1$, argument $r=60^\circ$.

Since the conditions are both necessary and sufficient, they are the required conditions.

II. Solution by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Let z_1, z_2, z_3 represent the vertices of a triangle in a z-complex-number-

plane, and let z'_1, z'_2, z'_3 represent the corresponding vertices in a z'-complex-number-plane.

A transformation can always be found that will change z_1 in z'_1, and z_2 into z'_2. For this it is sufficient that a and b in

$$\left.\begin{array}{l} z'_1=az+b \\ z'_2=az_2+b \end{array}\right\}\ldots.(1),$$

shall satisfy the condition

$$a=\frac{z'_2-z'_1}{z_2-z_1}, \text{ and } b=\frac{z'_1z_2-z'_2z_1}{z_2-z_1}.$$

Every transformation of the plane into itself that leaves every figure in the plane similar to itself is expressible in the form

$$z'=az+b\ldots\ (2).$$

The value of a and b in (1) must satisfy the equation $z'_3=az_3+b$. For this the necessary and sufficient condition is

$$\frac{z'_3-z'_1}{z_3-z_1}=\frac{z'_2-z'_1}{z_2-z_1};\ \therefore \begin{vmatrix} z'_1, & z_1, & 1 \\ z'_2, & z_2, & 1 \\ z'_3, & z_3, & 1 \end{vmatrix}=0.$$

The conditions that the sides shall be equal are

$$\frac{z_1-z_2}{z_2-z_3}=\frac{z_2-z_3}{z_3-z_1}=\frac{z_3-z_1}{z_1-z_2};\ \therefore \begin{vmatrix} z_1, & z_2, & 1 \\ z_2, & z_3, & 1 \\ z_3, & z_1, & 1 \end{vmatrix}=0.$$

Also demonstrated by *WILLIAM HOOVER*, *G. B. M. ZERR*, *J. W. YOUNG*, and *J. SCHEFFER*.

CALCULUS.

108. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

The hypotenuse of a plane right triangle increases uniformly at the rate of 1-12 of an inch a second. If the legs are as 2 to 3, at what rate is the area of the triangle increasing when the perpendicular from the right angle upon the hypotenuse is 12 inches ?

Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, O.; H. C. WHITAKER, Ph. D., Manual Training School, Philadelphia, Pa.; J. W. YOUNG, Cornell University, Ithaca, N. Y.; P. S. BERG, Larimore, N. D.; and D. G. DORRANCE, Jr., Camden, N. Y.

Let x=the length of the hypotenuse, and y=the area.

Then $dx=\frac{1}{12}$, and dy is required when the perpendicular from the right angle to the hypotenuse equals 12.

From the condition of the problem $y=\dfrac{3x^2}{13}$, whence $dy=\dfrac{6xdx}{13}$.

Now when the perpendicular from the right angle to the hypotenuse is 12 the hypotenuse must be 26. Hence, substituting in the last equation we have $dy=\dfrac{6\times26\times\frac{1}{12}}{13}=1$; *i. e.*, the area, at the time mentioned, is increasing at the rate of 1 square inch a second.

Also solved by *L. B. FILLMAN*, *ALOIS F. KOVARIK*, *J. SCHEFFER*, *C. D. SCHMITT*, and *G. B. M. ZERR*.

109. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the curve in which the product of the perpendiculars drawn from two fixed points to any tangent is constant.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn., and the PROPOSER.

Let the equation of the tangent be

$$y-y'=\frac{dy'}{dx'}(x-x').$$

And let the two points be $(a, 0)$ and $(-a, 0)$.

The product of the two perpendiculars is easily found to be

$$\frac{(xdy-ydx)^2-(ady)^2}{(dx)^2+(dy)^2}\text{ which}=b^2,$$

$$\text{or }\frac{(xp-y)^2-(ap)^2}{1+p^2}=b^2,\text{ or }(xp-y)^2=p^2(a^2+b^2)+b^2,$$

$$\text{or }y=px\pm\sqrt{[p^2(a^2+b^2)+b^2]}.$$

This is Clairaut's form, so that we have

$$y=mx\pm\sqrt{[m^2(a^2+b^2)+b^2]},$$

which is the well-known tangent to an ellipse.

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

MECHANICS.

108. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Can it be shown, as a result of Kepler's third law, that the distances are inversely proportional to the squares of the velocities?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

This can be demonstrated for circular orbits as follows:

$$t^2:t_1^2=a^3:a_1^3;\text{ but } t^2=4\pi^2a^2/v^2,\ t_1^2=4\pi^2a_1^2/v_1^2.$$

$\therefore\ 1/v^2:1/v_1^2=a:a_1$.

For elliptic orbits

$$v^2=\frac{\mu}{b}\left(\frac{2b-r}{r}\right);\ v_1^2=\frac{\mu}{b_1}\left(\frac{2b_1-R}{R}\right)$$

where b, b_1 are the semi-major axes, respectively.

$$\therefore\ \frac{1}{v^2}:\frac{1}{v_1^2}=\frac{br}{2b-r}:\frac{b_1R}{2b_1-R}=r:R \text{ when } r=R=\infty.$$

$\therefore$ The theorem is true for parabolic orbits.

109. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

If the sun were moved into the center of the earth's orbit, how much would the present length of the year be changed?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

It is presumed in this solution that the earth's orbit remains the same.

Let t=the present length=1 year; T=the required length; μ=absolute force; and a=semi-major axis.

$$\therefore\ t=2\pi\sqrt{\frac{a^3}{\mu}},\ T=\frac{2\pi}{\sqrt{\mu}}.\quad \therefore\ \frac{t}{T}=a\sqrt{a} \text{ or } T=\frac{t}{a\sqrt{a}}.$$

But $a=1$, and $t=1$ year.

$\therefore\ T=1$ year and the length of the year would remain the same.

110. Proposed by W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A tight roll of very thin and perfectly flexible oilcloth is placed upon a rough inclined plane, a portion of the cloth being unrolled, and, extending from underneath the roll, is spread out smoothly upon the inclined plane below. The roll is then allowed to descend under the action of gravity, picking up the cloth as it goes. Determine the motion as far as possible.

No solution of this problem has been received.

111. Proposed by WALTER H. DRANE, Graduate Student, Harvard Umvercity, Cambridge, Mass.

A man, weighing 150 pounds at the surface of the earth, ascends in a balloon until the area visible to him is $2\pi R^2(1-\frac{1}{2}\sqrt{2})$. What is his weight at that height?

Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, O.; P. S. BERG, B. Sc., Larimore, N. D.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

If R=the radius of the earth, and h=the altitude of the zone visible, then

$$2\pi Rh=2\pi R^2(1-\tfrac{1}{2}\sqrt{2}),\text{ whence } h=R(1-\tfrac{1}{2}\sqrt{2}).$$

Now if H be the distance of the balloonist from the earth's center, then

$$H:R::R:R-h \text{ or } H=R\sqrt{2}.$$

Now by the law of gravitation, $(R\sqrt{2})^2:R^2::150:x$, the required weight, that is, $x=\dfrac{150R^2}{2R^2}=75$.

112. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Vary the radius of curvature of a plane curve inversely as the abscissa; then the solution will give you, (1) Ryan's Equation of the Elastic Curve, and (2) Wood's Equation of the Hydrostatic Curve.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Radius of curvature $=\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=\dfrac{a^2}{2x}$.

$\therefore\ a^2(d^2y/dx^2)=2x[1+(dy/dx)^2]^{\frac{3}{2}}$. Let $dy/dx=p$.

$\therefore\ a^2(dp/dx)=2x(1+p^2)^{\frac{3}{2}}$ or $a^2p/\sqrt{(1+p^2)}=(x^2+A)$.

$\therefore\ p=dy/dx=\pm\sqrt{(x^2+A)}/\sqrt{[a^4-(x^2+A)^2]}$. Let $x^2+A=a^2\cos\theta$.

$$\therefore\ y=\mp\tfrac{1}{2}a^2\int\frac{\cos\theta d\theta}{\sqrt{(a^2\cos\theta-A)}}=\mp\tfrac{1}{2}a^2\int\frac{(1-2\sin^2\frac{1}{2}\theta)d\theta}{\sqrt{(a^2+1-A-2\sin^2\frac{1}{2}\theta)}}.$$

Let $\frac{1}{2}\theta=\varphi$ and $2/(a^2+1-A)=e^2$.

$$\therefore\ y=\pm\frac{a^2e^2}{2}\int\frac{(1-2\sin^2\varphi)d\varphi}{\sqrt{(1-e^2\sin^2\varphi)}}.$$

$\therefore\ y=\pm a^2E(e,\ \varphi)\mp(\frac{1}{2}a^2)(2-e^2)F(e,\ \varphi)+B\ldots.(1)$.

Since $\varphi=\frac{1}{2}\theta$, $y=\pm a^2E(e,\ \frac{1}{2}\theta)\mp(\frac{1}{2}a^2)(2-e^2)F(e,\ \frac{1}{2}\theta)+B\ldots.(2)$.

(1) represents Ryan's Elastic Curve and (2) represents the Hydrostatic Curve. In the above $e<1$ and $A<a^2+1$.

In the second equation the curve can never cross the line of force since $\sin(\frac{1}{2}\theta)$ cannot equal $1/e$.

Also solved by *J. SCHEFFER*.

DIOPHANTINE ANALYSIS.

58. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find a square proper fraction which if subtracted from unity will leave for remainder a square proper fraction.

I. Solution by C. E. ARMENTROUT, Professor of Mathematics, Rockingham Military Institute. Mt. Crawford, Va.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and P. S. BERG, Larimore, N. D.

Let $(y/x)^2$ be the square proper fraction.

$1-(y/x)^2=(n/m)^2$ or $m^2(x^2-y^2)=n^2x^2$.

Let $x=p^2+q^2$, $y=p^2-q^2$.

$\therefore\ 1-\left(\dfrac{p^2-q^2}{p^2+q^2}\right)=\dfrac{4p^2q^2}{(p^2+q^2)^2}$, where p can be any value greater than q.

Let $p=2$, $q=1$. $\therefore\ 1-(\frac{3}{5})^2=(\frac{4}{5})^2$.

Let $p=3$, $q=2$. $\therefore\ 1-(\frac{5}{13})^2=(\frac{12}{13})^2$.

II. Solution by M. A. GRUBER. A. M., War Department, Washington, D. C., and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.

This reduces to the simple statement that $a^2/c^2+b^2/c^2=1$, or that $a^2+b^2=c^2$. This is evidently satisfied in the case of every rational right triangle.

Thus, $3^2+4^2=5^2$, or $\frac{9}{25}+\frac{16}{25}=1$.

This is the same as saying $1-\frac{9}{25}=\frac{16}{25}$, thus fulfilling the condition of the problem.

Similarly an indefinite number of examples might be given.

Also solved by *J. H. DRUMMOND*, *ALOIS F. KOVARIK*, *J. W. YOUNG*, and the late *SYLVESTER ROBINS*.

86. Proposed by A. H. BELL, Hillsboro, Ill.

The edges of a rectangular parallelopiped are within one of the proportion 3:6:7, and if they are $3x$, $6x\mp1$, $7x$, then $(3x)^2+(6x\mp1)^2+(7x)^2=$the diagonal squared$=94x^2\mp12x+1=\square$. To find four integral values for x in this equation.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $94x^2\mp12x+1=(mx\mp1)^2=m^2x^2\mp2mx+1$.

$\therefore\ x=\pm\dfrac{2(m-6)}{94-m^2}$.

Expanding $\sqrt{94}$ as a continued fraction we get

$$\sqrt{94}=9+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9+}$$

The successive convergents are as follows :

$$\frac{9}{1},\quad \frac{10}{1},\quad \frac{29}{3},\quad \frac{97}{10},\quad \frac{126}{13},\quad \frac{223}{23},\quad \frac{1241}{128},\quad \frac{1644}{151},\quad \frac{12953}{1336},\quad \frac{14417}{1487},$$

$$\frac{85038}{8771},\quad \frac{99455}{10258},\quad \frac{184493}{19029},\quad \frac{652934}{67345},\quad \frac{1490361}{1537191},\quad \frac{2143295}{221064},\quad \cdots\cdots$$

The first integral value of x is given by substituting for m the eighth convergent, the second by using the sixteenth convergent for m.

Let $m=\frac{1464}{151}$, $x=84258$, $3x=252774$, $6x-1=505547$, $7x=589806$, $94x^2-12x+1=(816911)^2$.

Let $m=-\frac{1464}{151}$, $x=357870$, $3x=1073610$, $6x+1=2147221$, $7x=2505090$, $94x^2+12x+1=(3469679)^2$.

Let $m=\frac{2143295}{221064}$, then $x=361179226608$, $3x=1083537679824$, $6x-1=2167075359647$, $7x=2528254586256$, $94x^2-12x+1=(3501762523489)^2$.

Let $m=-\frac{2143295}{221064}$, then $x=1534042236912$, $3x=4602126710736$, $6x+1=9204253421471$, $7x=10738295658384$, $94x^2+12x+1=(14873091304609)^2$.

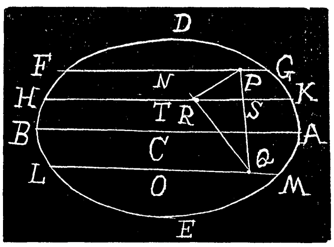
AVERAGE AND PROBABILITY.

95. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three random points are taken in an ellipse, one on each side of the major axis, and the third anywhere in the ellipse. Find the average area of the triangle formed by joining the three points.

Solution by the PROPOSER.

Let $ADBE$ be the ellipse, AB the major and DE the minor axis, C the center, and PQR the triangle formed by joining the random points. Through P, Q, R draw the chords FG, LM, HK parallel to AB.



It is only necessary to consider the relative positions in which the chord HK lies between the chords FG and LM.

Let $CD=CE=b$, $CA=CB=a$, $CN=u$, $CO=v$, $CT=w$, $NP=x$, $OQ=y$, $TR=z$, $TS=t$, $NG=x'$, $OM=y'$, $TK=z'$.

Then $x'=(a/b)\sqrt{(b^2-u^2)}$, $y'=(a/b)\sqrt{(b^2-v^2)}$, $z'=(a/b)\sqrt{(b^2-w^2)}$, $t=y-\dfrac{(y-x)\,(w-v)}{u-v}$.

Area $PQR=\frac{1}{2}(t-z)(u-v)=A$, $t>z$. Area $PQR=\frac{1}{2}(z-t)(u-v)=A_1$, $t<z$.

The limits of u are 0 and b ; of v, $-b$ and 0 ; of w, v and u ; of x, $-x'$ and x'; of y, $-y'$ and y'; of z, $-z'$ and t, and t and z'.

Hence, since the whole number of ways the three points can be taken is $\frac{1}{4}\pi^3a^3b^3$ and doubling since P and Q are interchangeable the required average area is

$$\Delta=\frac{8}{\pi^3a^3b^3}\int_0^b\int_{-x'}^{x'}\int_{-b}^{0}\int_{-y'}^{y'}\int_v^u\left[\int_{-z'}^{t}A\,dz+\int_t^{z'}A_1\,dz\right]dudxdvdydw$$

$$=\frac{4}{\pi^3a^3b^3}\int_0^b\int_{-x'}^{x'}\int_{-b}^0\int_{-y'}^{y'}\int_v^u\left[\frac{a^2}{b^2}(b^2-w^2)+\left(y-\frac{(y-x)(w-v)}{u-v}\right)^2\right]$$

$$(u-v)dudxdvdydw$$

$$=\frac{4}{3\pi^3a^3b^3}\int_0^b\int_{-x'}^{x'}\int_{-b}^0\int_{-y'}^{y'}[3a^2-(a^2/b^2)(u^2+uv+v^2)+y^2+xy+x^2]$$

$$(u-v)^2dudxdvdy$$

$$=\frac{4}{9\pi^3a^2b^4}\int_0^b\int_{-x'}^{x'}\int_{-b}^0\left[20a^2+6x^2-\frac{6a^2u^2}{b^2}-\frac{6a^2uv}{b^2}-\frac{8a^2v^2}{b^2}\right]$$

$$\cdot\sqrt{(b^2-v^2)}(u-v)^2dudxdv$$

$$=\frac{2}{36\pi^3a^2b^4}\int_0^b\int_{-x'}^{x'}(39\pi a^2b^2u^2+8\pi a^2b^4+12\pi b^2u^2x^2-12\pi a^2u^4+3\pi b^4x^2$$

$$+96a^2b^3u-16a^2bu^3+32b^3ux^2)dudx$$

$$=\frac{2a}{54\pi^3b^5}\int_0^b(126\pi b^2u^2-48\pi u^4+27\pi b^4+320b^3u-80bu^3)\sqrt{(b^2-u^2)}du$$

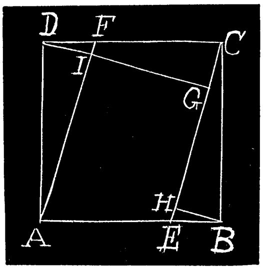
$$=\left(\frac{35}{72\pi}+\frac{32}{9\pi^3}\right)ab.$$

96. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A random straight line is drawn across a square; find the chance that it intersects two opposite sides. [From *Byerly's Integral Calculus*, page 209].

Solution by the PROPOSER.

Let $ABCD$ be the given square whose side is a. Draw the parallel lines AF and CE. Then all lines parallel to AF and lying between AF and CE will intersect two opposite sides of the square. The total number of such lines is proportional to IG. The total number of lines parallel to AF and crossing the square is proportional to $DG+BH$.



Hence, the probability of a line drawn parallel to AF and intersecting the opposite sides of the square is $p_0=\dfrac{IG}{DF+BH}$.

Let the angle $FAB=\theta$. Then $DG=CD\sin\theta=a\sin\theta$, $IG=DG-DI=DC\sin\theta-AD\cos\theta=a(\sin\theta-\cos\theta)$, and $BH=EB\sin\theta=BC\cot\theta\sin\theta=a\cos\theta$.

$$\therefore p_0=\frac{a(\sin\theta-\cos\theta)}{a(\sin\theta+\cos\theta)}.$$

$$\therefore\ p=\frac{a[(\sin\theta'-\cos\theta')+(\sin\theta''-\cos\theta'')+(\sin\theta'''-\cos\theta''')+\text{ etc.}]}{a[(\sin\theta'+\cos\theta')+(\sin\theta''+\cos\theta'')+(\sin\theta'''+\cos\theta''')+\text{ etc.}]}$$

$$=\frac{a\int(\sin\theta-\cos\theta)d\theta}{a\int(\sin\theta+\cos\theta)}.$$

The limits of θ for favorable cases are $\frac{1}{2}\pi$ and $\frac{1}{4}\pi$ and doubled, for the total cases $\frac{1}{2}\pi$ and 0.

$$\therefore\ p=\frac{2\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}[\sin\theta-\cos\theta]d\theta}{\int_0^{\frac{1}{2}\pi}[\sin\theta+\cos\theta]d\theta}=(\sqrt{2}-1).$$

NOTE.—The answer given in Professor Byerly's Integral Calculus, Edition ot 1890, is $\frac{1}{2}-1/\pi\log 2$.

This is the answer to the following problem: *From a point taken at random in the side of a square, a line is drawn at random across the square. What is the chance that the line will intersect the opposite side of the square?*

Professor Zerr interpreted the problem in this way and solved it accordingly.

Solutions somewhat similar to the one above were received from J. M. Colaw and Lon C. Walker.

Professor Walker sent a solution of No. 93. His answer is $\frac{1}{24}\pi p^2$. This difference in results is due to the different interpretations of the problem by Professors Zerr and Walker.

97. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A straight line is drawn at random across a circle, and five points are taken at random in the surface of the circle. Required the chance that all the points are on the same side of the line.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let θ=angle subtended by random chord at the center. Then area of the smaller segment is $a^2(\theta-\sin\theta\cos\theta)$. Hence the required chance is

$$p=\frac{\int_0^\pi 2[a^2(\theta-\sin\theta\cos\theta)]^5 a\sin\theta d\theta}{\int_0^\pi (\pi a^2)^5 a\sin\theta d\theta}$$

$$=\frac{1}{\pi^5}\int_0^\pi (\theta-\sin\theta\cos\theta)^5\sin\theta d\theta=1-\frac{128}{9\pi^2}+\frac{2768896}{33075\pi^4}=.4184.$$

MISCELLANEOUS.

89. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

Find the earth's average density and mass, having given that the attraction of a ball of lead 2 feet in diameter, on a particle placed close to its surface, is less than the earth's attraction in the ratio 1:10250000, and that the density of lead is $11\frac{1}{2}$ times that of water.

Solution by the PROPOSER.

Regard the earth as a sphere, radius 3956 miles.

$3956\times5280=20887680$ feet.

Let D be the earth's average density, then since the attractions at their surfaces are proportional to their densities multiplied by their radii, we get at once, $1:10250000=11.5\times1:D\times20887680$.

$\therefore\ D=5.643$.

The mass of the earth from these data is $\frac{4}{3}\pi(20887680)^3\times5.643\times62.5$ pounds$=m$.

$\therefore\ m=1477.3374(20887680)^3$ pounds$=738.6687(2088768)^3$ tons.

Also solved by *J. SCHEFFER.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

142. **Proposed by M. J. CRAWFORD, Principal of Crawford's Academy, Savannah, Ga.**

A gentleman has a garden 400 feet long and 300 feet wide, which he wishes to raise 9 inches higher by means of the earth to be dug out of a ditch 6 feet wide and surrounding the entire garden. How deep must the ditch be?

143. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.**

A's income$=a/b$th part$=\frac{3}{4}$ of B's income. A's outgo$=m/n$th part$=\frac{1}{2}$ of B's income. B's outgo$=p/q$th part$=1/1$ of A's income. What is the ratio of their savings?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than June 10.

ALGEBRA.

134. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.**

Solve neatly and briefly the equations

$x^3+x^2y+y^3=53\ldots.(1)$, $y^3+y^2z+z^3=13\ldots.(2)$, and $z^3+z^2x+x^3=31\ldots.(3)$.

135. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Tangents parallel to the three sides are drawn to the in-circle. If p, q, r, be the lengths of the parts of the tangents within the triangle, prove that

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

GEOMETRY.

163. Proposed by J. C. NAGLE, Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

Given the equal sides of an isosceles triangle and the radius of the inscribed circle to solve the triangle. As a numerical example let the known sides be 27 and the radius of the inscribed circle 3.5. The problem occured in connection with some mill work and the exterior angles of the triangle were required in order to make patterns for iron braces.

164. Proposed by J. M. HARCOURT, M. D., 305 Clinton Street, Brooklyn, N. Y.

Given two tangents to a parabola, find the locus of the center of the nine-point circle of the triangle by the two given tangents and any third tangent.

165. Proposed by W. H. ECHOLS, B.Sc., C.E., Professor of Mathematics, University of Virginia, Charlottsville, Va.

$OB=b$, $OA=a$ are the semi-conjugate diameters of an ellipse. Draw BM perpendicular to and equal to OA, cutting it in N. Show that as M slides on the fixed line OM and N on OA the point B traces the curve.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

CALCULUS.

128. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The differential equation of a curve is $\frac{d^3y}{dx^3}+y=0$. Find its equation, there being the additional conditions that for $x=0$, $y=1$, that the tangent at the point (0, 1) makes an angle of 45° with the axes, and finally that that point is a point of inflexion.

129. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Among all quadrilaterals inscribed in an ellipse, to determine that which contains the greatest area.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

MECHANICS.

120. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

I project an elastic particle along a chord c of a smooth fixed circular rim of diameter d. The coefficient of elasticity between the particle and the rim is e, and the particle continually rebounds. Find the length of the chord described after the nth rebound.

121. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Two equal scale pans of mass m hang at rest over a smooth pulley. An inelastic particle, mass M, is dropped from a height h into one pan, and simultaneously another of equal mass and elasticity e is dropped from the same height into the other. Prove that every impact occurs when the pans are in their original positions, and find the total space described by either pan before motion ceases.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

AVERAGE AND PROBABILITY.

104. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

In a given sphere two radii are drawn at random, and a point taken in each at random. (1) Find the chance that the distance between the two points does not exceed the radius of the sphere. (2) Find the distance between them.

105. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the average distance of the center of an ellipsoid, axes $2a$, $2b$, and $2c$, from its surface.

106. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Required the average distance between two points in a hemisphere.

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

MISCELLANEOUS.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If the refractive index of a medium at any point be $\mu = x$, prove that the path of the ray will be the curve $\frac{2x}{a} = \frac{c}{a} l^{y/a} + \frac{a}{c} l^{-(y/a)}$, a and c being constants.

106. Proposed by J. W. YOUNG, Oliver Graduate Student in Mathematics, Cornell University, Ithaca, N. Y.

Prove that $\frac{(2m)!}{(m!)^2}$ is an integer; and more generally that $\frac{(nm)!}{(m!)^n}$ is an integer, m, n being any positive integers.

107. Proposed by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The index of refraction of a medium varying inversely as the square root of the distance, prove that the path of a ray of light in the medium is a cycloid.

*** Solutions of these problems should be sent to J. M. Colaw not later than June 10.

NOTES.

Dr. G. A. Miller, of Cornell University, has been appointed to an issistant professorship of mathematics at Leland Stanford Jr. University.

Miss R. H. Vivian, Alumnæ Fellow in Mathematics at the University of Pennsylvania, has been appointed Instructor in Mathematics at Wellesley.

Professor D. E. Smith, principal of the Brockport, N. Y., State Normal School, has been called to a professorship of mathematics in Teachers College, Columbia University.

Dr. J. M. Rice died at Northboro, Mass., on March 2d, aged sixty-eight years. He was appointed Professor of Mathematics at the Naval Academy at Annapolis in 1870. In 1879, in colaboration with Professor W. W. Johnson, appeared their Treatise on Differential Calculus,—a meritorious work.

Alexander Macfarlane, M. A., D. Sc., LL. D., delivered six lectures entitled "British Mathematicians of the Nineteenth Century," in the Chemical Laboratory of Lehigh University, beginning April 12, 1901. The life and work of the following mathematicians were presented in the order and at the time named below : George Peacock (1791-1858), April 12th, 11:30 a. m.; Augustus DeMorgan (1806-1871), April 13th, 11:30 a. m.; Sir William Rowan Hamilton (1805-1865), April 16th, 4:00 p. m., George Boole (1815-1864), April 19th, 11:30 a. m.; Arthur Cayley (1821-1895), April 20th, 11:30 a. m.; William Kingdon Clifford (1845-1879), April 23rd, 4:00 p. m.

BOOKS AND PERIODICALS.

Teachers' Manual of School Arithmetic. By J. M. Colaw, A. M., Assistant Editor of THE AMERICAN MATHEMATICAL MONTHLY, Montery, Va., and John K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pa., author of Table Book and Test Problems in Mathematics. 8vo. cloth, 301 pages. Richmond, Va.: B. F. Johnson Publishing Co.

The first fourteen pages of this Manual is devoted to a brief history of Arithmetic, which is of great interest and value. Pages 19 to 51 are devoted to the teaching of Arithmetic; in which valuable suggestions to teachers are set forth and the best methods of teaching the subject are presented. The remaining part of the book contains excellent solutions of the problems in the authors' School Arithmetic and some solutions of special problems. B. F. F.

On Continuity and Irrational Numbers and the Nature and Meaning of Numbers. By Richard Dedekind. Translated by Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan. 8vo, red cloth, 115 pages. Price, 75 cents. Chicago : The Open Court Publishing Co.

The lovers of Mathematics who are only familiar with the English language are again placed under great obligation to The Open Court Publishing Company and to Professor Beman for bringing out this translation of Dedekind's Memoirs on Continuity and Irrational Numbers, and the Nature and Meaning of Numbers. Every teacher of Mathematics who has not read these Memoirs in the original ought to buy this little book and read it carefully several times. B. F. F

Plane and Solid Analytical Geometry. An Elementary Text-Book. By Charles H. Ashton, A. M., Instructor in Mathematics in Harvard University. 8vo, cloth and leather back, xiii+266 pages. Price, $1.00. New York: Charles Scribner's Sons.

This book is intended for classroom use and is designed to meet the needs of classes having from sixty to seventy hours to devote to the subject. The conics are treated from their ratio definitions,—a method that commends itself to most teachers. Numerous well selected problems are inserted at the end of nearly every chapter. Two chapters are devoted to loci, and Poles and Polars are treated by the aid of harmonic divisions. The treatment of Analytic Geometry of Space is commendable, and Quadratic Surfaces here receive brief consideration. B. F. F.

A Manual of Laboratory Physics. By H. M. Tory, M. A., Lecturer in Mathematics, McGill University, Montreal; late Demonstrator of Physics, McDonald Physics Building, McGill University, and F. H. Pitcher, M. Sc., A. M. I. E. E., late Demonstrator of Physics, McDonald Physics Building, McGill University, Montreal. Crown 8vo., cloth, ix+288 pages. Price, $2.00. New York: John Wiley & Sons.

This book presents an elementary Laboratory Course in Sound, Light, Heat, Magnetism, and Electricity. Each experiment is carefully described. Under "Apparatus Required" is given an exact statement of the apparatus necessary for the particular experiment. Under "Theory of Experiment" is set forth the theory involved in the special experiment under consideration. Under "Practical Directions" are given such directions as a director would give if standing beside the student. In addition, a tabulated example of the observations and results are added as a guide to the student. A corresponding "Blank" to be filled by the student is added, and thus is provided a permanent record of his work. We consider this Manual meritorious in every particular, B. F. F.

Annals of Mathematics. Published under the auspices of Harvard University. Price, $2.00 per year in advance. Second Series, Vol. II., No. 3.

The April number contains the following articles: "Sufficient Conditions in the Calculus of Variations," by Professor W. F. Osgood; Lagrange's "Equation in the Calculus of Variations, and The Extension of a Theorem," by J. K. Whittemore; "On Some Points in the Theory of the Hypergeometric Function, Expressed as a Double Circuit Integral," by R. M. Hathaway; "A Theory of Continued Fractions," by Dr. Derrick N. Lehmer; "Note on the Dual of a Focal Property of the Inscribed Ellipse," by Professor R. E. Allardice; "A Simplified Solution of the Cubic," by Dr. Emory McClintock.
B. F. F.

Le Mathematiche, Vol. I., No. 2.

All American contributions to this journal should be sent to Dr. George Bruce Halsted, University of Texas, Austin, Texas. B. F. F.

ERRATA.

Page 26, line 13, for "Wolfgan" read Wolfgang.
Page 27, line 21, for "Emporer" read Emperor.
Page 28, line 14, for "*Varein*" read *Verein*.
Page 28, line 2 from bottom, for "Derichlet" read Dirichlet.
Pagc 29, line 20, for "elleptic" read elliptic.
Page 29, line 26, for "corporium" read corporum.
Page 30, line 13, for "michanique" read mecanique.
Page 31, line 4, for "imaginery" read imaginary.
Page 56, line 10, for "Homogenious" read Homogeneous.
Page 56, line 11, for "Quarternary" read Quaternary.

The above errors were pointed out to us by W. J. Greenstreet, Editor of *The Mathematical Gazette.* We shall appreciate the kindness if others will follow Professor Greenstreet's example, as we wish to rectify all errors. Also the following are noticed:

Page 28, line 2 from bottom, for "Vorlesunger" read Vorlesungen.
Problem 130, page 54, next to last line, after "$x^n-y^n=a$," insert $x-y=b$.

THE AMERICAN MATHEMATICAL MONTHLY.

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VOL. VIII. MAY, 1901. No. 5.

BIOGRAPHY.

FRANZ SCHMIDT.

BY DR. GEORGE BRUCE HALSTED.

On the fifteenth of December, 1902, will be fitly celebrated in Hungary the centenary of the birth of Bolyai János, but the man through whom the world come to possession of this truest genius will not be present.

On the seventh of March, 1901, at Budapest, in his seventy-fifth year, died the architect Franz Schmidt, whose name will ever be justly connected with that of John Bolyai.

Until my investigations in 1896 at Budapest, Kolozsvár and Maros-Vásárhely, what was known of Bolyai came wholly from a brief article by Franz Schmidt, then living at Temesvár, entitled "Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya," Grunert's Archiv, Bd. 48, 1868, page 217. This was translated into Italian by Angelo Forti under the title: "Intorno alla vita ed agli scritti di Wolfgang e Giovanni Bolyai di Bolya matematici ungheresi," and published in the Bulletino di Bibliografia e di storia delle scienze matematiche et fisiche. T. I. Rom, 1868.

But Schmidt had written the article at the prompting of Hoüel, who translated it into French and published it first in the Mémoires de la Société des sciences physiques et naturelles de Bordeaux, Tome V, Bordeaux 1867. He afterward prefixed it to his translation of John Bolyai's Science Absolute of Space in 1868.

FRANZ SCHMIDT.

In a "Note du Traducteur" Hoüel says: "The name of Bolyai has become inseparable from these profound discoveries. It was therefore a duty for European science to draw this name from an unjust oblivion.

The author of the Notice of which we publish the translation, M. Fr. Schmidt, has consecrated himself to this work of reparation with an indefatigable devotion not discouraged either by the slowness of communication between the different parts of the Hungarian empire, or by the difficulty of procuring, even in the native state of Bolyai, the necessary details about the man who has rendered it celebrated.

It is also to the learned architect of Temesvár that I owe the possession of the excessively rare work of John Bolyai, which I have reproduced after the Notice of its author, certain that it cannot fail to excite the interest of all those who truly love science.

By this publication, la Société des Sciences physiques et naturelles continues the work which it commenced by inserting in its preceding volume the researches of Lobachevski on the same subject. May this homage rendered to the Hungarian geometer decide his compatriots to produce from his papers, deposited in the College of Maros-Vásárhely, the remarkable works which they must contain, and of which we know as yet only the titles."

It was from this same copy of the Latin text sent from Temesvár to Hoüel by Schmidt, sent to Schmidt by Professor Samuel Szabó, that in 1868 G. Battaglini made the Italian version published by him in his Giornale di Matematiche.

Schmidt begins his paper as follows [I translate from a copy he presented me while I was with him at Budapest in his peaceful home on the banks of the beautiful blue Danube, at Rudolf-rakpart 8]: "The biographical sketch that I here trace is still very incomplete; but as there seems little likelihood of the appearance soon of a detailed life-history, I publish what I have been able to learn from information printed, oral or manuscript, in the hope that any who are in position to add to or make more precise these data, may be induced soon to complete what is here given." A quarter of a century passed away before the world knew anything further of John Bolyai.

In June, 1891, appeared my translation: "THE SCIENCE ABSOLUTE OF SPACE *Independent of the Truth or Falsity of Euclid's Axiom XI (which can never be decided a priori)*. By John Bolyai. In the introduction to the first edition of this I say: "Beyond the *Appendix*, whose translation into English is here given for the first time, John Bolyai published nothing; and the thousand pages of manuscript which he left have never been read by a competent mathematician. They are in the library of the Reformed College of Maros-Vásárhely. What discoveries might lie hidden in his papers!" So I formed the project of making the journey from Austin to Maros-Vásárhely, and entered into correspondence with Professar Koncz, a pupil of John Bolyai's father, who succeeded to his professorial chair and residence.

Professor Koncz wrote me in Latin, and finally sent me precious documents which I planned to incorporate in a more extensive Life of Bolyai. On

February 14th, 1895, he sent me a French translation of the letter written in Magyar by John to his father from Temesvár, on November 3d, 1823, which fixes the date of John Bolyai's creation of the non-Euclidean geometry. I gave an English translation of this wonderful letter in the Introduction to the fourth edition of my "Science Absolute."

At last in July, 1896, in Budapest on my way to Maros-Vásárhely, I met and sojourned with Franz Schmidt himself, who long had been my correspondent. Many things he told me. For example, how first he came to know of John Bolyai. "My father," said he, "Anton Schmidt, was architect in Temesvár, 1817—1860, during which period he had often to build military structures. In later years he often told me of an officer of engineers, a Siebenburger (a Transylvanian, that is a Magyar from Erdély) with whom he always feared to come into contact. He related how this Magyar exhibited, to demonstrate the strength of his arm and the firmness of his Damascus sword, to each of his visitors, how he cut off with a single stroke stout iron nails which had been driven into the door post. That was our Bolyai János."

Franz Schmidt had visited Maros-Vásárhely in 1893 to find the grave of János, and arrange for the erection of a monument over it. He said: "John's grave was. even up to 1894, wholly unrecognizable, without any mark. Only his then still living nurse, Juliana Szöcs, who had attended upon him for eight years and who buried him, could show me the place." In the journal "Közérdek," published at Maros-Vásárhely, on November 11th, 1894, is the following note: "John Bolyai's grave for thirty-four years was without any mark, like to the ground. In 1893 Arehitect Franz Schmidt visited Maros-Vásárhely. To his indefatigable zeal we owe it, that the Mathematico-Physical Society of Budapest, by a subscription, erected to the author of the Appendix a pyramid of trachyte."

On my arrival at Maros-Vásáhely I had scarcely closed the door of my room at the "Transylvania" when in rushed a handsome old Magyar gentleman, threw his arms around me and kissed me! This was Professor Koncz Jözsef. The great chest containing John Bolyai's papers was opened for me. I was particularly struck with his writings on the Theory of Imaginaries, where he far surpassed Gauss, explaining the square root of minus one ($\sqrt{-1}$), the i of Gauss, as a new unit, for a new set of numbers qualitatively different from the numbers previously used, but no more imaginary than fractions or negative numbers. This is the view to which the world gradually attained unhelped by Bolgai's genius, for his work thereon, though sent to Leipzig to contest for a prize in 1838, had never been published. Long before seeing it I had suggested as a name for the new unit the word Neomon, giving neomic numbers in place of the unfortunately named imaginary numbers.

Even more was I struck with a manuscript treatise entitled "Raumlehre." Perhaps my enthusiasm was contagious. That year (1896) began a new epoch in the world's knowledge of Bolyai. That it was still Franz Schmidt will appear from the following excerpt which I translate from an article by Stäckel and

Engel, Math. Annalen, B. 49, pp. 149—206, 1897. "That we now find ourselves in more favorable position, is due before all to the persistence and devotion of architect Franz Schmidt of Budapest, who for thirty years continuously has worked to elucidate and expound the part of the two Bolyais in the history of the non-Euclidean geometry.

In December, 1896, the Royal Society of Sciences of Goettingen graciously put at his disposal a copy of the Correspondence between Wolfgang Bolgai and Gauss, and in concert with him one of us published in the Goettingen Nachrichten a selection of the mathematical part of this Correspondence.

But we owe still more to Architect Schmidt. In collaboration with his son, Prof. Dr. Martin Schmidt of Pressburg, he has subjected the papers left by the two Bolyais to a new inspection and has sent to us a series of communications about the results. But especially has he made accessible to us certain writings hitherto unknown, written in the Magyar language."

These Magyar documents are those into possession of which I had come years before, for a wonder independently of Franz Schmidt, and had translated, sending in August, 1895, a translation to Maros-Vásárhely to Professor Koncz József for him to review and annotate, This he did. But though I treasure his nine folio pages of annotation, I could not make up my mind to publish my translation simply because it was almost wholly about the father, Bolyai Farkas, while it is the son, Bolyai János, upon whom the world's interest should be centered. I wrote this to Professor Koncz in 1895 and begged for documents about János, for a picture of János, for any notes or remarks or anything pertaining to the immortal János, the most perfect case of genius in the world's history. On September 28th, 1895, he made a most precious and splendid response, as a few sentences from his letter will show. He says :

"I have the honor to send you : 1. Biographic data about Wolfgang Bolyai, written by John Bolyai. N. B. The lines underlined in red are not "conformes à la verité."

2. Biographic data on John Bolyai written by Coleman Szily, first Secretary of the Hungarian Academy of Sciences, from the notes of Gregory Bolyai.

3. The will of John Bolyai and his signatures at different epochs.

4. The photograph of Wolfgang Bolyai, taken on his death bed. N. B. There does not exist any portrait of John Bolyai.

5. An extract from the studies of John Bolyai in 1818.

I have not been able to procure any details about the duels of John Bolyai."

By the munificence of the Hungarian Academy of Science, Franz Schmidt was able in 1899 in conjunction with Paul Stäckel to publish the entire Bolyai-Gauss correspondence in a beautiful quarto of 208 pages. He had in 1897 at his own expense issued the first translation of the Science Absolute in Magyar, fulfilling thereby, as he says, a wish cherished for thirty years.

At the Bolyai János celebration next December, entwined in honor with the genius of the master must be the devotion of Franz Schmidt.

KINETIC DERIVATION OF TANGENT EQUATION.

By A. LATHAM BAKER, University of Rochester, Rochester, N. Y.

The following investigation is, I am afraid, more interesting than valuable, though it gives a very simple way of getting the equation of a tangent. But there is evidently an underlying principle which I have not succeeded in getting at, which would, if found, undoubtedly be valuable.

Since a curve can be considered as generated by the motion of a point, we can consider this point as the tracing point of a mechanism so adjusted as to trace the proper curve. This tracing point can be imagined to be the common point of various sets of x and y arms, each set composed of a certain number of x points, or of y points.

Now if, for example, the mechanism be set for the curve represented by the equation $ax^2+by^2=c$, and we stop the action of an x and a y point, leaving an x and a y point current, the new curve will be a straight line in *continuation* of the old motion. That is, the straight line will be a tangent line.

The algebraic equivalent of this would be found by writing

$$axx_1+byy_1=c$$

which would consequently be the *equation of the tangent line*, found by stopping one pair of coördinates (indicated by the subscripts) and letting the remaining pair remain current.

In the same way, from the algebraic mechanism

$$ax^m+by^m=c$$

we get the equation for the tangent line

$$ax_1{}^{m-1}x+by_1{}^{m-1}y=c.$$

Extending this thought to more complicated forms, we have the following divisions of algebraic mechanisms, using the word *simple* to indicate sets o terms containing only x's or only y's, and *compound* for terms containing both x's and y's.

A. TERMS SIMPLE AND COMPOSED OF THE SAME NUMBER OF TRACING POINTS, that is of the same degree, as in the examples already given, $ax^2+by^2=c$. In these we adopt a *current pair* of coördinates and make all the others fixed, viz., $ax_1x+by_1y=c$.

Example. $x+3y=a$. To have a fixed point (point of tangency) we must have two mechanisms compounded, viz.,

$$x+3y+x+3y=2a,$$

whence, leaving one x and one y current, we get $x+3y+x_1+3y_1=2a$.

B. TERMS SIMPLE, BUT WITH AN UNEQUAL NUMBER OF TRACING POINTS, with the subdivisions as to variable terms.

B_1. TWO TERMS ONLY, example $y=x^2$.

B_2. MORE THAN TWO TERMS BUT EACH TERM BALANCED BY ITS MATE IN THE OTHER LETTER.

B_3. MORE THAN TWO TERMS BUT THE TERMS NOT BALANCED.

B_1. As in A, we make one pair of tracing points (coördinates) current, but here the potency of the tracing point is inversely proportional to the number required to constitute a term, and the number of terms from which a current y point must be taken to balance a current x point must be *directly* proportional to the number of points in the term (degree of the term). Thus in $y=x^3+b$, one y must be balanced by three x's. Hence, taking three mechanisms together, we get

$$y+2y=3x^3+3b.$$

Now having current one y and three x's, we get $y+2y_1=3xx_1^2+3b$.

Similarly in $y^2=x^3$, two y's must be balanced by three x's and we must take $2y^2+y^2=3x^3$.

Leaving current two y's and three x's, we get $2yy_1+y_1^2=3xx_1^2$.

Again in $y^2=4px$, one x balances two y's, hence $2y^2=4px+4px$. Leaving current one x and two y's we get $2yy_1=4px+4px_1$.

B_2. Here we put each set of homogeneous terms equal to a summand of the absolute term, operate on these equations by $A-B_1$, and add the results.

Example. $y+y^2=x+x^2+c$.

$$\begin{array}{ll} y-x=p & y+y_1-(x+x_1)=2p \\ y^2-x^2=q & 2yy_1-2xx_1=2q \end{array}$$

whence $y+y_1+2yy_1=x+x_1+2xx_1+2c$.

B_3. Here we take the sum of the y's (or x's) and break it up into the sum of term of degree similar to the original, each summand being put equal to a term of the x's (or y's), and apply $A-B_2$. Or take the sets of homogeneous terms and place them equal to summands of the constant term; operate on these summands by $A-B_2$ and add the results.

Example. $y+y^2=x$ $\quad y=p \quad y^2=q$

Whence $y+y_1=p+p_1$

$2yy_1=q+q_1$. Whence $y+y_1+2yy_1=x+x_1$.

Example. $y+y^2=x^3$. $\quad$ $y=p^3$, $\quad y+2y_1=3pp_1^2$

$y^2=q^3$, $\quad 2yy_1+y_1^2=3qq_1^2$

Whence $y+2y_1+2yy_1+y_1^2=3xx_1^2$.

Example. $y=x^3-x^2+x$. $\quad$ $p=x^3$, $\quad p+p_1=3xx_1^2-p_1$

$q=-x^2$, $\quad q+q_1=-2xx_1$

$r=x$, $\quad r+r_1=x+x_1$

Whence $y+y_1=3xx_1^2-2xx_1+x+x_1-x_1^3$.

Example. $y^2=x^3-x^2$.

$$\begin{array}{rl} p^2=x^3, & 2pp_1+p_1^2=3xx_1 \\ q^2=-x^2, & 2qq_1=-2xx_1 \\ & q_1^2=-x_1^2 \end{array}$$

Whence $2yy_1+y_1^2=3xx_1^2-2xx_1-x_1^3$. Here q_1^2 is added so that we can sum up to y's.

Example. $y+y^2=x^3$.

$$\begin{array}{rl} y=p, & y+y_1=2p \\ y^2=q, & 2yy_1=2q \\ -x^3=r, & x_1^3-3xx_1^2=2r \end{array}$$

Whence $y+y_1+2yy_1+x_1^3-3xx_1^2=0$.

Example. $y+y^2=x^3$.

$$\begin{array}{rl} y=p, & y+2y_1=3p \\ y^2=q, & y_1^2+2yy_1=3q \\ -x^3=r, & -3xx_1^2=3r \end{array}$$

Whence $y+2y_1+y_1^2+2yy_1-3xx_1^2=0$.

Example. $y=x^3-x^2$.

$$\begin{array}{rl} p=x^3, & p+p_1=3xx_1^2-x_1^3 \\ q=-x^2, & q+q_1=-2xx_1 \end{array}$$

Whence $y+y_1=3xx_1^2-x_1^3-2xx_1$.

Example. $y+y_2=-x^2$.

$$\begin{array}{rl} y=-p^2, & y+y_1=-2pp_1 \\ y^2=-q^2, & 2yy_1=-2qq_1 \end{array}$$

Whence $y+y_1+2yy_1=-2xx_1$.

Example. $y+y^2=x^3-x^2+x$.

$$\begin{array}{rl} p+p^2=x^3, & p+2p_1+2pp_1+p_1^2=3xx_1^2 \\ q+q^2=-x^2, & q+q_1+2qq_1=-2xx_1 \\ r+r^2=x, & r+2rr_1+r_1=x+x_1 \end{array}$$

Whence $y+2yy_1+y_1=3xx_1^2-2xx_1+x+x_1-p_1-p_1^2=3xx_1^2-2xx_1+x-x_1^3+x_1^2$.

Example. $y+y^2=x^3-x^2+x$.

$$\begin{array}{rl} y-x=p, & y+y_1-x-x_1=2p \\ y^2+x^2=q, & 2(yy_1+xx_1)=2q \\ -x^3=r, & -3xx_1^2-x_1^3=2r \end{array}$$

Whence $y+y_1-x-x_1+2yy_1+2xx_1=3xx_1^2-x_1^3$.

The value for $2r$ is found as follows. Suppose r variable say z, then $z+2z=-3x^3$, $z+2z_1=-3xx_1^2$, but once z is really constant this becomes $2z_1=2r=-3xx_1^2-x_1^3$.

C. Terms compound, but of equal degree.

C_1. The x's and y's balanced.

C_2. The x's and y's not balanced.

C_1: As in A make one pair current, taking each element from a different term.

Example. $xy=1$. $xy+xy=2$, so that a current y may be balanced by a current x and leave a fixed point. Whence $xy_1+x_1y=2$.

Example. $x^2y^2=c$. $x^2y^2+x^2y^2=2c$. Whence $xx_1y_1^2+yy_1x_1^2=2c$.

Example. $x^3y^3=c$. $x^3y^3+x^3y^3=2c$. Whence $x_1^3y_1^2y+xx_1^2y_1^3=3c$.

C_2. As in B_3, break up the absolute term into summands, each summand being equal to one of the class of terms, and proceed as in B_1, making one pair current, selecting its constituents from a number of terms inversely proportional to the weight of the elements selected, operating upon the terms as classified in $A-C_1$.

Example. $x^2y^3=c$. Here two x's must be balanced by three y's. Hence $2xx_1y_1^3+3yy_1^2x_1^2=5c$.

Example. $x^2y+xy^2=a$. $\quad x^2y=a_1, \quad 2xx_1y_1+x_1^2y=3a_1$.
$\quad xy^2=a_2, \quad 2x_1y_1y+xy_1^2=3a_2$.

Whence $x(2x_1y_1+y_1^2)+y(2x_1y_1+x_1^2)=3a$.

D. TERMS COMPOUND AND NOT OF THE SAME DEGREE.

D_1. x'S AND y'S NOT BALANCED IN EACH TERM.

D_2. x'S AND y'S BALANCED IN EACH TERM.

D_1. Decompose into summands as in B_3, C_1 and add the results.

Example. $x^2y+x^2y^3=a$. $\quad x^2y=p, \quad x_1^2y+2x_1y_1x=3p$
$\quad x^2y^3=q, \quad 3x_1^2y_1^2+2x_1y_1^3x=5q$

Whence $x_1^2y+2x_1y_1x+3x_1^2y_1^2y+2x_1y_1^3x=3a+2q=3a+2x_1^2y_1^3$.

Example. $x^2y+x^3y=c$. $\quad x^2y=p, \quad x_1^2y+2xx_1y_1=3p$
$\quad x^3y=q, \quad x_1^3+3xx_1^2y_1=4q$

Whence $x_1^3y+3xx_1^2y_1+x_1^2y+2xx_1y_1=3c+x_1^3y_1$.

Example. $x^3y+xy=c$. $\quad x^3y=p, \quad x_1^3y+3xx_1^2y_1=4p$
$\quad xy=q, \quad x_1y+y_1x=2q$

Whence $x_1^3y+3xx_1^2y_1+x_1y+y_1x=2c+2p=2c+2x_1^3y_1$.

D_2. In D_1 owing to the unequal weight of x and y in the terms we are compelled by C_2 to take a number of terms proportional to the degree of the term before we can operate. In D_2 however, owing to the cancellations we must specifically designate this requirement or it will be overlooked. Hence, treat each term by itself and combine the results, each multiplied by the degree of its term.

Example. $xy+x^2y^2=a$. $\quad x_1y+xy_1+2(x_1^2y_1y+xx_1y_1^2)=2p+4q=2a+2x_1^2y_1^2$.

Example. $x^4y^4+xy=a$. $\quad x^4y^4=p, \quad 4(x_1^4y_1^3y+y_1^4x_1^3x)=8p$
$\quad xy=q, \quad x_1y+xy_1=2q$

Whence $4x_1^4y_1^3y+4xy_1^4x_1^3+x_1y+xy_1=8p+2q=2a+6x_1^4y_1^4$.

Example. $xy+x^3y^3=a$. $\quad xy=p, \quad x_1y+xy_1=2p$
$\quad x^3y^3=q, \quad 3(xx_1^2y_1^3+x_1^3y_1^2y)=3.2q$

Whence $x_1y+y_1x+3xx_1^2y_1^3+3x_1^3y_1^2y=2p+6q=2a+4x_1y_1+4x_1^3y_1^3$.

Example. $x^2y^2+x^3y^3=a$. $\quad 2(x_1xy_1^2+x_1^2yy_1)+3(xx_1^2y_1^3+x_1^3yy_1^2)=4p+6q=4a+2x_1^3y_1^3$.

E. COMPOSITES OF *A*, *B*, *C*, *D*.

Make each homogeneous set of terms equal to a summand of the absolute term. Operate on these by $A-D$, and add the results.

Example. $xy+x+y=a$. $\qquad xy=p, \qquad x_1y+xy_1=2p$

$\qquad x+y=q, \qquad x+x_1+y+y_1=2q$

Whence $x_1y+xy_1+x+y+x_1+y_1=2x_1y_1+2x_1+2y_1$, and $x_1y+xy_1+x+y-x_1y_1=a$.

Example. $xy+x=1$. $\qquad xy=p, \qquad xy_1+x_1y=2p$

$\qquad x=q, \qquad x+x_1=2q$

Whence $x_1y+xy_1+x+x_1=2(p+q)=2$.

Example. $x^2+y^2+xy=1$. $\qquad x^2+y^2=p, \qquad 2(xx_1+yy_1)=2p$

$\qquad xy=q, \qquad xy_1+x_1y=2q$

Whence $xy_1+x_1y+2xx_1+2yy_1=2$.

Example. $x^2+xy=a$. $\qquad x^2=p, \qquad 2xx_1=2p$

$\qquad xy=q, \qquad xy_1+x_1y=2q$

Whence $x_1y+xy_1+2xx_1=2a$.

PROOF THAT FOR MAXIMUM CURRENT THE EXTERNAL AND INTERNAL RESISTANCES SHOULD BE EQUAL.

By JAMES S. STEVENS, Professor of Physics, University of Maine, Orono, Me.

If we have a cells to connect we may take m series with n cells in each series. Then $mn=a$.

By formula for Ohm's law,

$$C=\frac{nE}{\frac{nr}{m}+R}$$

where r and R are respectively the internal resistance of each cell and the total external resistance.

Dividing numerator and denominator by n we have

$$C=\frac{E}{\frac{r}{m}+\frac{R}{n}}.$$

For maximum current it is necessary to make $\frac{r}{m}+\frac{R}{n}$ a minimum.

The expression takes the following form :

$$\frac{R}{n}+\frac{r}{a/n}, \quad \frac{aR+rn^2}{an}.$$

Placing the first differential coefficient of this expression equal to zero we have

$$\frac{2an^2rdn-a^2Rdn-an^2rdn}{a^2n^2}=0.$$

From which

$$rn^2=aR, \quad n^2=\frac{aR}{r}.$$

Replacing the value of a/m for one factor in n^2,

$$n\frac{a}{m}=\frac{aR}{r}, \quad \frac{n}{m}=\frac{R}{r}, \quad R=\frac{nr}{m}.$$

Or the external resistance equals the total internal resistance. This is seen to be a minimum value for the expression differentiated since the value of the second differential coefficient is greater than zero for the positive value of n, —the only value it can have.

THE RADIUS OF THE TERRESTRIAL SPHEROID.

By F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

If there be nothing *new* under the sun, it may not be uninteresting to expand the *old*.

Represent the earth's equatorial radius by a, the geographical latitude by ϕ, and the geocentric latitude by ϕ'; then since $x^2/a^2+y^2/b^2=1$, we have $\tan\phi=-dx/dy$, and $\tan\phi'=y/x$. Also, since $b^2=a^2(1-e^2)$, we have

$$y^2=a^2(1-e^2)-(1-e^2)x^2 \text{ and } y/x=(1-e^2)\tan\phi.$$

$$\therefore\ x=\frac{a\cos\phi}{\sqrt{(1-e^2\sin^2\phi)}} \text{ and } y=\frac{a(1-e^2)\sin\phi}{\sqrt{(1-e^2\sin^2\phi)}} \ldots \ (1).$$

Now, the radius of the terrestrial spheroid for any latitude ϕ, is $\rho=\sqrt{(x^2+y^2)}$.

$$\therefore\ \rho=a\sqrt{\left(1-\frac{e^2(1-e^2)\sin^2\phi}{1-e^2\sin^2\phi}\right)},=a\sqrt{[1-e^2(1-e^2)(\sin^2\phi+e^2\sin^4\phi)]}.$$

By assuming $e^2=1-f^2$ and $\sin^2\phi=\frac{1}{2}(1-\cos2\phi)$, Encke obtains the series

$$\log\rho = 9.9992747 + 0.0007271\cos 2\phi - 0.0000018\cos 4\phi,$$

in which the equatorial radius is unity.

Making $x = \rho\cos\phi'$ and $y = \rho\sin\phi'$, then the former equation of (1) may be written $e^2 = \dfrac{\rho^2\cos^2\phi' - a^2\cos^2\phi}{\rho^2\sin^2\phi\cos^2\phi'}$; and by means of this value of e^2, the elimination of e^2 from the latter equation of (1) may be effected.

$$\therefore\ \rho^2\sin^2\phi' = \frac{(\rho^2\sin^2\phi\cos^2\phi' - \rho^2\cos^2\phi' + a^2\cos^2\phi)^2}{\rho^2\sin^2\phi\cos^2\phi\cos^2\phi'}.$$

$$\therefore\ \rho = a\sqrt{\left(\frac{\cos\phi}{\cos\phi'\cos(\phi - \phi')}\right)} \ldots . (a).$$

Formula (a) may be deduced in another way; by assuming that

$$e\sin\phi = \sin\psi \ldots . (2),$$

we obtain

$$\rho\sin\phi' = a(1 - e^2)\sin\phi\sec\psi \ldots (\alpha), \text{ and } \rho\cos\phi' = a\cos\phi\sec\psi \ldots . (\beta).$$

From (α) and (β) by easy deductions,

$$\rho\sin(\phi - \phi') = \tfrac{1}{2}ae^2\sin 2\phi\sec\psi \ldots . (\alpha'), \text{ and } \rho\cos(\phi - \phi') = a\cos\psi \ldots . (\beta').$$

From (2) and (β') we have, respectively,

$$e = \sin\psi/\sin\phi \text{ and } \cos\psi = \rho\cos(\phi - \phi')/a;$$

and after transforming (α'), etc., we obtain

$$\rho\sin(\phi - \phi'), = a \times \frac{\cos\phi}{\sin\phi} \times \frac{1 - \cos^2\psi}{\cos\psi}, = a \times \frac{\cos\phi}{\sin\phi} \times \frac{a^2 - \rho^2\cos^2(\phi - \phi')}{a\rho\cos(\phi - \phi')}.$$

$$\therefore\ \rho^2 = \frac{a^2\cos\phi}{\sin(\phi - \phi')\cos(\phi - \phi')\sin\phi + \cos^2(\phi - \phi')\cos\phi}.$$

Expanding this denominator, combining terms, etc., we have (a) by a second method of reduction.

In order to obtain formula (a) by a third method, we remember that $a^2/b^2 = \tan\phi/\tan\phi'$ and that $x^2 + (a^2/b^2)y^2 = a^2$; or after obvious transformations,

$$\rho^2\cos^2\phi' + (a^2/b^2)\rho^2\sin^2\phi' = a^2, \text{ or } \left[\cos^2\phi' + \left(\frac{\sin\phi \times \cos\phi'}{\cos\phi \times \sin\phi'}\right)\sin^2\phi'\right]\rho^2 = a^2,$$

from which formula (a) is readily deduced.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

140. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah Iowa.

$\frac{1}{7}=0.\dot{1}4285\dot{7}$; $\frac{1}{14}=0.0\dot{7}1428\dot{7}$; $\frac{1}{21}=0.0\dot{0}4761\dot{9}$. Notice that the sum of the figures in each period is equal to 27. This is not true with $\frac{1}{72}$, $\frac{1}{73}$. Is there any general law of which these are special cases, and if so, what is it?

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

When the *period* of a circulating decimal fraction consists of an *even* number of figures, the second half of that period can be found by subtracting each figure of the first half from the figure 9. In this case the sum of the figures constituting a period is always divisible by 9, because we have as many 9's as there are figures in a half-period. Should the half-period have 3 or 6 or 9 or 12, etc., figures, then the sum of the figures in the period will be divisible by 27, and only then. The period of $\frac{1}{72}$ is only 8 and cannot be divisible by 27. The period of $\frac{1}{73}$ consists of eight figures of which the first four are 0, 1, 3, 6, consequently the last four are 9, 8, 6, 3, and the sum of the figures $4\times 9=36$.

141. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

If the alloy in a half-dollar be 1-13th of the mass, and the coin be worth a cent if it be all alloy, what should be the exact value of the coin if it be all pure silver?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics.The Temple College, Philadelphia, Pa.; P. S. BERG, Larimore, N. D.; J. SCHEFFER, A. M., Hagerstown, Md.; and HARVEY M. DAVIS, Brown University, Providence, R. I.

Suppose the half-dollar worth 50 cents.

Then since alloy is worth $\frac{1}{13}$ of a cent, $\frac{12}{13}$ of the mass=silver, is worth $49\frac{12}{13}$ cents.

$\frac{1}{13}$ silver is worth $\frac{1}{12}$ of $49\frac{12}{13}$.

$\frac{13}{13}$ silver is worth $\frac{13}{12}$ of $49\frac{12}{13}$ cents$=54\frac{1}{12}$ cents.

GEOMETRY.

49. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics a Springfield, Mo.

Given a conic and two circumscribing triangles of the conic; prove that the six vertices of the triangles are con-conic.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

This is most neatly solved, perhaps, by the theory of projection.

Projecting two of the vertices of the first triangle into the circular points

at infinity, the third vertex is the focus of the parabola into which the given conic becomes by projection.

The projection of the second triangle is a circumscribed triangle to the parabola, the circle circumscribing which triangle passing through the focus of the parabola, and proving the theorem.

The reciprocal theorem is: Two triangles are inscribed in a conic; their six sides touch another conic.

150. Proposed by **WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Find the equation to a sphere cutting orthogonally four given spheres.

Solution by the PROPOSER.

Let $x^2+y^2+z^2+2Ax+By+Cz+D=0\ldots.(1)$, be the sphere cutting orthogonally the spheres

$$x^2+y^2+z^2+2a_1x+2b_1y+2c_1z+d_1=0\ldots.(2),$$
$$\Sigma x^2+\Sigma 2a_2x+d_2=0\ldots.(3),$$
$$\Sigma x^2+\Sigma 2a_3x+d_3=0\ldots.(4),$$
$$\Sigma x^2+\Sigma 2a_4x+d_4=0\ldots.(5),$$

Now, two spheres cut each other orthogonally if their radii and the distance between their centers form a right triangle; this requires, for (1) and (2),

$$(A-a_1)^2+(B-b_1)^2+(C-c_1)^2=A^2+B^2+C^2-D+a_1^2+b_1^2+c_1^2-d_1,$$
$$\text{or, } 2Aa_1+2Bb_1+2Cc_1-D-d_1=0\ldots.(6).$$

Similarly for the intersection of (1) and each of (3), (4) and (5),

$$2Aa_2+2Bb_2+2Cc_2-D-d_2=0\ldots.(7),$$
$$2Aa_3+2Bb_3+2Cc_3-D-d_3=0\ldots.(8),$$
$$2Aa_4+2Bb_4+2Cc_4-D-d_4=0\ldots.(9),$$

(6), (7), (8), and (9) give

$$\triangle A=\begin{vmatrix} 2b_1, & 2c_1, & -1, & d_1 \\ 2b_2, & 2c_2, & -1, & d_2 \\ 2b_3, & 2c_3, & -1, & d_3 \\ 2b_4, & 2c_4, & -1, & d_4 \end{vmatrix}\ldots.(10),$$

and like values for B, C, and D, which in (1) gives the required equation.

Similar demonstrations were received from *J. W. YOUNG*, *G. B. M. ZERR*, and *LON C. WALKER*.

MECHANICS.

113. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A rough uniform rod, length $2a$, is placed with a length $c(>a)$ projecting over the edge of the table. Prove that the rod will begin to slide over the edge when it has turned through an angle $\tan^{-1}\left[\frac{\mu a^2}{a^2+9(c-a)^2}\right]$. [From Loudon's *Rigid Dynamics.*]

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let ϕ be the angle through which the rod has rotated at the end of time t from the beginning of motion, μ=the coefficient of friction, W=the weight of the rod, and R=the normal reaction between the table and rod.

Taking moments about the point of contact of the table and rod,

$$\frac{W}{g}[\tfrac{1}{3}a^2+(c-a)^2]\frac{d^2\phi}{dt^2}=W(c-a)\cos\phi\ldots.(1);$$

and resolving parallel and perpendicular to to the rod,

$$\mu R=W\sin\phi+\frac{W}{g}(c-a)\,\frac{d\phi^2}{dt^2}\ldots.(2),$$

$$R=W\cos\phi-\frac{W}{g}(c-a)\,\frac{d^2\phi}{dt^2}\ldots.(3).$$

Multiplying (1) by $2\frac{d\phi}{dt}$ and integrating, and noticing that when $\phi=0$, $\frac{d\phi}{dt}=0$,

$$[\tfrac{1}{3}a^2+(c-a)^2]\frac{d\phi^2}{dt^2}=2g(c-a)\sin\phi\ldots.(4).$$

Solving (1) and (4) for $\frac{d^2\phi}{dt^2}$ and $\frac{d\phi^2}{dt^2}$, and substituting in (2) and (3),

$$\mu R=W\sin\phi\frac{a^2+9(c-a)^2}{a^2+3(c-a)^2}\ldots.(5),$$

$$R=W\cos\phi\frac{a^2}{a^2+3(c-a)^2}\ldots.(6).$$

Dividing (6) by (5),

$$\tan\varphi=\frac{\mu a^2}{a^2+9(c-a)^2}\ldots.(7).$$

Also solved by *G. R. DEAN*, and *G. B. M. ZERR.*

114. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Prove that the *inclination* of a perfectly rough inclined plane must be $\theta=\sin^{-1}[e^2/(2-e^2)]$, in order that an ellipse of minimum eccentricity e may be capable of resting in equilibrium on the plane.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The center of the ellipse will be vertically over the point of support ; also the vertical through the point of support and the parallel to the plane through the center will form conjugate diameters.

$\therefore$ The acute angle between these diameters$=\frac{1}{2}\pi-\theta$. Since the potential energy is greatest at a point bordering on motion, the major axis will bisect the angle $\frac{1}{2}\pi-\theta$; hence the major axis makes with the conjugate diameters angles $\frac{1}{4}\pi-\frac{1}{2}\theta$ and $\frac{1}{4}(3\pi)+\frac{1}{2}\theta$.

We have for conjugate diameters, then, this condition,

$$a^2\sin^2[\tfrac{1}{4}\pi-\tfrac{1}{2}\theta]\sin[\tfrac{1}{4}(3\pi)+\tfrac{1}{2}\theta]+b^2\cos[\tfrac{1}{4}\pi-\tfrac{1}{2}\theta]\cos[\tfrac{1}{4}(3\pi)+\tfrac{1}{2}\theta]=0,$$

$$\text{or } a^2\sin^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)-b^2\cos^2(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta)=0.$$

$$\therefore\ \frac{a^2-b^2}{a^2}=e^2=\frac{2\sin\theta}{1+\sin\theta}\ \text{ or }\ \sin\theta=\frac{e^2}{2-e^2}.$$

$$\therefore\ \theta=\sin^{-1}[e^2/(2-e^2)].$$

115. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A vessel in the shape of a parallelopiped, filled with water, has in its horizontal bottom a rectangular opening, whose dimensions are a and b, which is shut up by a slider. Supposing this slider to be opened with a uniform motion in the direction of a. To find the depth of the water in the vessel after the time T at the moment when the slider has passed through the space a, a denoting the horizontal section of the water in the vessel.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let h=height of parallelopiped, K=area of bottom, x=depth of water at any time, y=the distance the slider has opened at any time.

Then average area of orifice$=b\int_0^a ydy/\int_0^a dy=\frac{1}{2}ab$.

$$\therefore\ t=-\frac{2K}{ab\sqrt{2g}}\int\frac{dx}{\sqrt{x}}=-\frac{4K\sqrt{x}}{ab\sqrt{2g}}+C.$$

Since $x=h$ when $t=0$, $C=\frac{4K\sqrt{h}}{ab\sqrt{2g}}$.

$\therefore t=T=\frac{4K}{ab\sqrt{2g}}(\sqrt{h}-\sqrt{x})$. $\therefore x=\left(\frac{4K\sqrt{h}-Tab\sqrt{2g}}{4K}\right)^2$.

AVERAGE AND PROBABILITY.

98. Proposed by REV. PREBENDARY WHITWORTH, A. M.

A has £m and B has £n. They play for points until one of them has lost all his money. If α and β be the respective chances that A and B win any point, the expectation of the number of points played will be

$$\frac{n\alpha^n(\alpha^m-\beta^m)-m\beta^m(\alpha^n+\beta^n)}{(\alpha-\beta)(\alpha^{m+n}-\beta^{m+n})}.$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A_m=A's chance of winning, B_n=B's chance of winning.

Then nA_m, mB_n=A's and B's expectation, respectively.

$\therefore$ Expectation of number of points played$=E$.

Then $E=(nA_m-mB_n)/(\alpha-\beta)$.

Let A_x=A's chance when he has x pounds and B has $m+n-x$ pounds.

$\therefore A_x=\frac{\beta}{\alpha+\beta}A_{x-1}+\frac{\alpha}{\alpha+\beta}A_{x+1}$.

$A_x-A_{x-1}=(\alpha/\beta)(A_{x+1}-A_x)$. Giving x successive values from 1 to x we $A_1-A_0=(\alpha/\beta)(A_2-A_1)$, $A_2-A_1=(\alpha/\beta)(A_3-A_2)$, etc.

By continued multiplication we get $A_1-A_0=(\alpha/\beta)^{x-1}(A_x-A_{x-1})$ or $A_x-A_{x-1}=(\beta/\alpha)^{x-1}(A_1-A_0)$.

Give x successive values from 1 to x and add

$$A_x-A_0=(A_1-A_0)[1+\beta/\alpha+(\beta/\alpha)^2+\ldots\ldots+(\beta/\alpha)^{x-1}].$$

But $A_0=0$. $\therefore A_x=A_1[1-(\beta/\alpha)^x]/[1-(\beta/\alpha)]$.

$A_{m+n}=1$. $\therefore 1=A_1[\alpha^{m+n}-\beta^{m+n}]/[\alpha^{m+n-1}(\alpha-\beta)]$.

$\therefore A_1=[\alpha^{m+n-1}(\alpha-\beta)]/(\alpha^{m+n}-\beta^{m+n})$.

$\therefore A_x=[\alpha^{m+n-1}(\alpha^x-\beta^x)]/[\alpha^{x-1}(\alpha^{m+n}-\beta^{m+n})]$.

$\therefore A_m=[\alpha^n(\alpha^m-\beta^m)]/(\alpha^{m+n}-\beta^{m+n})$.

Similarly, $B_n=[\beta^m(\alpha^n-\beta^n)]/(\alpha^{m+n}-\beta^{m+n})$.

$\therefore E=\frac{n\alpha^n(\alpha^m-\beta^m)-m\beta^m(\alpha^n-\beta^n)}{(\alpha-\beta)(\alpha^{m+n}-\beta^{m+n})}$.

99. Proposed by E. B. SEITZ.

A point is taken at random in the surface of a given circle, and from it a line equal in length to the radius is drawn, so as to lie wholly in the surface of the circle. Find the chance that the line intersects in a given diameter. [No. 135, *The Analyst.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be the random point, and AB the given diameter.

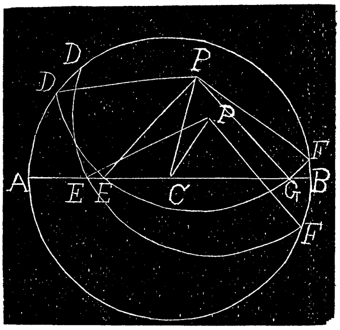
$CP=x$, $\angle PCB=\theta$. With P as center and radius$=r$, the radius of the given circle. Draw the arc DEF.

From $\theta=0$ to $\theta=\cos^{-1}\left(\frac{x}{2r}\right)$, F is below AB.

From $\theta=\cos^{-1}\left(\frac{x}{2r}\right)$ to $\theta=\frac{1}{2}\pi$, F is above AB.

Let $\cos^{-1}\left(\frac{x}{2r}\right)=\theta'$, $p=$the required chance. Then

$$p=\frac{\int_0^r\int_0^{\theta'}EF.xdxd\theta+\int_0^r\int_{\theta'}^{\frac{1}{2}\pi}EG.xdxd\theta}{\int_0^r\int_0^{\frac{1}{2}\pi}DF.xdxd\theta}.$$



Arc $EF=r[\theta-\sin^{-1}(\frac{x}{r}\sin\theta)+\cos^{-1}(\frac{x}{2r})]$, arc DF $=2\,r\cos^{-1}(\frac{x}{2r})$, arc $EG=r[\pi-2\sin^{-1}(\frac{x}{r}\sin\theta)]$.

$$\therefore\; p=\frac{\int_0^r\left(\int_0^{\theta'}EFd\theta+\int_{\theta'}^{\frac{1}{2}\pi}EGd\theta\right)xdx}{\int_0^r\int_0^{\frac{1}{2}\pi}2r\cos^{-1}\left(\frac{x}{2r}\right)xdxd\theta}$$

$$=\frac{12}{\pi r^2(4\pi-3\sqrt{3})}\int_0^r\left(\int_0^{\theta'}[\theta-\sin^{-1}(\frac{x}{r}\sin\theta)+\cos^{-1}(\frac{x}{2r})]d\theta\right.$$

$$\left.+\int_{\theta'}^{\frac{1}{2}\pi}[\pi-2\sin^{-1}(\frac{x}{r}\sin\theta)]d\theta\right)xdx=\frac{12}{\pi r^2(4\pi-3\sqrt{3})}\left[\int_0^r\int_0^{\theta'}[\theta-\sin^{-1}(\frac{x}{r}\sin\theta)\right.$$

$$+\cos^{-1}(\frac{x}{2r})]xdxd\theta+\int_0^r\int_0^{\frac{1}{2}\pi}[\pi-2\sin^{-1}(\frac{x}{r}\sin\theta)]xdxd\theta$$

$$\left.-\int_0^r\int_0^{\theta'}[\pi-2\sin^{-1}(\frac{x}{r}\sin\theta)]xdxd\theta\right]$$

$$=\frac{12}{\pi r^2(4\pi-3\sqrt{3})}\left[\int_0^r\int_0^{\theta'}[\theta-\pi+\cos^{-1}(\frac{x}{2r})]xdxd\theta+\pi\int_0^r\int_0^{\frac{1}{2}\pi}xdxd\theta\right.$$

$$\left.+\int_0^{\frac{1}{2}\pi}\int_0^{2r\cos\theta}\sin^{-1}(\frac{x}{r}\sin\theta)d\theta xdx-2\int_0^{\frac{1}{2}\pi}\int_0^r\sin^{-1}(\frac{x}{r}\sin\theta)d\theta xdx\right]$$

$$=\frac{12}{\pi r^2(4\pi-3\sqrt{3})}\Bigg[\int_0^r\Big(\tfrac{3}{2}(\cos^{-1}\frac{x}{2r})^2-\pi\cos^{-1}\frac{x}{2r}+\tfrac{1}{2}\pi^2\Big)xdx$$

$$+\int_0^{\frac{1}{3}\pi}(4r^2\theta\cos^2\theta+\tfrac{1}{2}r^2\cot\theta\cos2\theta-\tfrac{1}{2}r^2\theta\text{cosec}^2\theta)d\theta$$

$$-2\int_0^{\frac{1}{2}\pi}(\tfrac{1}{2}r^2\theta+\tfrac{1}{4}r^2\cot\theta-\tfrac{1}{4}r^2\theta\text{cosec}^2\theta)d\theta\Bigg].$$

$$\therefore\ p=\frac{7\pi^2+8\pi\sqrt{3}-54}{3\pi(4\pi-3\sqrt{3})}.$$

NOTE.—This is problem 135 of *The Analyst*. Quite a discussion arose at the time of its first appearance, in which such eminent mathematicians of Prof. Benj. Pierce, of Harvard, took part. An incomplete solution by Professor Heaton was published. We believe Professor Zerr's solution is correct. ED. F.

100. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Required the average distance between two points in opposite sides of a regular $2n$-gon.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and the PROPOSER.

From any point P at a distance x from the vertex A, in the side $AB=a$ of the regular polygon, draw PE perpendicular to the opposite side DC.

Let Q be any point in DC between D and E at the distance$=y$ from E. Then we have

$$PQ=\sqrt{a^2\cot^2\frac{\pi}{2n}+y^2}\ .$$

The average distance of all points in DE from the point P will be the same as the required average when the line PE is made to move parallel to itself the entire length of a side of the polygon.

The elements of AB and CD at the points P and Q are dx and dy, respectively. Therefore the required mean is

$$M=\frac{\int_0^a\int_0^x\sqrt{a^2\cot^2\frac{\pi}{2n}+y^2}\,dxdy}{\int_0^a\int_0^x dxdy}$$

$$=\frac{2}{a^2}\int_0^a\int_0^x\sqrt{a^2\cot^2\frac{\pi}{2n}+y^2}\,dxdy=\frac{1}{a^2}\int_0^a\Bigg[x\sqrt{a^2\cot^2\frac{\pi}{2n}+x^2}$$

$$+a^2\cot^2\frac{\pi}{2n}\log\left(\frac{x+\sqrt{a^2\cot^2\frac{\pi}{2n}+x^2}}{a\cot\frac{\pi}{2n}}\right)\Bigg]dx=\Bigg[\tfrac{1}{3}\text{cosec}\frac{\pi}{2n}-\tfrac{2}{3}\tan\frac{\pi}{4n}\cot^2\frac{\pi}{2n}$$

$$+\cot^2\frac{\pi}{2n}\log\left(\frac{1+\sin\frac{\pi}{2n}}{\cos\frac{\pi}{2n}}\right)\Big].$$

COROLLARY. If $a=1$, $n=2$, then

$$M=\frac{2-\sqrt{2}}{3}+\log(1+\sqrt{2}).$$

Also solved by *G. B. M. ZERR*, and *HENRY HEATON*.

MISCELLANEOUS.

90. Proposed by DR. E. D. ROE, Jr., Syracuse University, Syracuse, N. Y.

I shot my rifle at different ranges and found the following table of elevations e, for the vernier peep sight, for the given distances s:

s	e
0	21.0
100	24 5
200	28.5
300	33.5
400	40.0
500	48 5

The distances are measured in yards. How shall a table of elevations be constructed, giving the arguments e, for every five yards up to 500 yards? Do not give the whole table, but explain the method, and illustrate by giving a computation, carrying the result to three places of decimals. An actual problem.

Solution by the PROPOSER.

The solution of this problem, which is doubtless somewhat arbitrary, was as follows: We have partly given and partly implied in the data, the following scheme:

s	0	100	200	300	400	500
e	21.0	24.5	28.5	33.5	40.0	48.5
Δe	3.5	4.0	5.0	6.5	8.5	
$\Delta^2 e$	0.5	1.0	1.5	2 0		
$\Delta^3 e$	0.5	0.5	0.5			
$\Delta^4 e$	0	0				

In this we notice that the fourth and third differences are constant, while the second are in arithmetical progression. Assuming that we have discovered the law, we may extend the second, third and fourth differences up to the column under 400, and thus have the complete data for interpolation by means of the method of finite differences. According to this method, if e_s denote the elevation for the distance s, we have,

$$e_{y\times100+(x/20)\times100}=e_{100y+5x}=e_{100y}+\frac{x}{20}\frac{\Delta e}{1\,!}100y+\frac{x}{20}\left(\frac{x}{20}-1\right)\frac{\Delta^2 e}{2\,!}100y$$

$$+\frac{x}{20}\left(\frac{x}{20}-1\right)\left(\frac{x}{20}-2\right)\frac{\Delta^3 e_{100y}}{2\,!},$$

where $y=0, 1, 2, 3, 4$.

$x=1, 2, \ldots 19$. As the complete solution, and giving the 95 values sought.

Example: It is required to find e_{425}. Here $y=4$, $x=5$, and $e_{400}=40$, $\Delta e_{400}=8.5$, $\Delta^2 e_{400}=2.5$, $\Delta^3 e_{400}=0.5$, $\Delta^4 e_{400}=0$. Hence

$$e_{425}=40+\frac{1}{4}\,8.5+\frac{1}{4}\left(\frac{1}{4}-1\right)\frac{2.5}{2\,!}+\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)\frac{0.5}{3\,!}$$

$$=40+2.125-0.234+0.027=41.918.$$

Remark: With a table like this satisfactory results in hunting may be obtained. The table is contained in a small note book. But the style of hunting must be changed. Longer distances must be used, and the work resolves itself into judging distances, and variations in the wind, setting elevation and wind guage sights, accurate sighting and firm arm holding of the rifle.

Also solved by *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

144. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A hired a house for one year for $300; at the end of four months he takes in M as a partner; and at the end of eight months he takes in P. At the end of the year what rent must each pay? [From Greenleaf's *National Arithmetic*, page 442.]

145. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

By discounting a note at 20% per annum, I get $22\frac{1}{2}$% per annum interest; how long does the note run? [From Ray's *Higher Arithmetic*, page 405.]

*** Solutions of these problems should be sent to B. F. Finkel not later than July 10.

ALGEBRA.

136. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $a^{x^2}b^{y^2}=c\ldots\ldots(1)$, and $c^{x+y}=ab\ldots\ldots(2)$.

137. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve, if possible, $a^x+b^x=c$.

138. Proposed by **HARRY S. VANDIVER**, Bala, Pa.

Show that the number of solutions in positive integers for x, y, and z of $x^3+2y^3+4z^3-6xyz=1$ is infinite.

$*_*^*$ Solutions of these problems should be sent to J. M. Colaw not later than July 10.

GEOMETRY.

166. Proposed by **S. F. NORRIS**, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to bo 25,000 miles? [From Wentworth's *New Plane and Solid Geometry*, page 381, No. 619.] Required solution by Geometry.

167. Proposed by **JOHN J. QUINN**, Professor of Mathematics, High School, Warren, Pa.

If at the vertex of an isosceles triangle one of whose basal vertices is pivoted and the other free to move in a straight line. a rhombus be pivoted with sides parallel to the sides of the triangle, the locus of every point on the rhombus except the one which is its intersection with the fixed side of the triangle is an ellipse.

168. Proposed by **MISS GUBELMAN**, Student Southern Illinois State University, Carbondale, Ill.

To draw a perpendicular to one side of a triangle dividing it into two equivalent parts.

$*_*^*$ Solutions of these problems should be sent to B. F. Finkel not later than July 10.

CALCULUS.

130. Proposed by **J. SCHEFFER**, A. M., Hagerstown, Md.

Solve the differential equation $x^x(\frac{dy}{dx}+y\log x)-a=0$.

131. Proposed by **F. P. MATZ**, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College Mechanicsburg, Pa.

Integrate $2/x$, with regard to $d[\sqrt{(1-x^2)}]$.

132. Proposed by **JOHN M. COLAW**, A. M., Monterey, Va.

What expression derived from the *polar* equation of a curve is equivalent to the expression for dy/dx derived from the *Cartesian* equation of the same curve? Prove work with $\rho=2r\cos\theta$.

$*_*^*$ Solutions of these problems should be sent to J. M. Colaw not later than July 10.

MECHANICS.

122. Proposed by **B. F. FINKEL**, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that a pressure P applied uniformly to a solid in all directions will reduce its dimensions along three perpendicular axes in ratio $1:1+p-2q$, p being the elongation along one face and q the contraction along the other. [Barker's *Physics*.]

123. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.**

Two equal uniform rods AB, BC are freely hinged at B; C rests on a rough horizontal plane, and A is attached to a point above it. When C is as far as possible from A for equilibrium, AB, BC make angles α, β, respectively, with the vertical. Find the coefficient of friction between the rod at C and the plane.

*** Solutions of these problems should be sent to B. F. Finkel not later than July 10.

AVERAGE AND PROBABILITY.

107. **Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.**

Two points are taken at random in the curved surface of a hemisphere. Show (1) that the average length of the straight therein is $\frac{32r}{9\pi}$; and (2) that the average length of the arc of a great circle, which joins them, is $\frac{4r}{\pi}$.

108. **Proposed by A. H. HOLMES, Brunswick, Me.**

Required the average area of the quadrilateral whose sides are a, b, c, and d.

109. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

A cylinder pierces a sphere in such a manner that the cylinder is tangent internally to the projection of the sphere in the plane xy. Find (1) the average surface, (2) the average volume of the sphere included within the cylinder.

*** Solutions of these problems should be sent to B. F. Finkel not later than July 10.

MISCELLANEOUS.

108. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

To divide the arc of a cardioid into eight equal parts.

109. **Proposed by J. SCHEFFER, A. M., Hagerstown, Md.**

Find the latitude of the place where the sun's centre remains above the horizon for a hundred successive days.

In problem 105, Miscellaneous, April Number, Vol. VIII, No. 4,

$$\frac{2x}{a}=\frac{c}{a}l^{y/a}+\frac{a}{c}l^{-y/a} \text{ should be } \frac{2x}{a}=\frac{c}{a}e^{y/a}+\frac{a}{c}e^{-y/a}.$$

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

NOTES.

Mr. H. W. Kuhn, of Cornell University, has been appointed Professor of Mathematics in the Ohio State University at Columbus, Ohio.

Dr. George Bruce Halsted has been invited to make at the Denver meeting of the Association for the Advancement of Science, August 24-31, 1901, a supplementary report on Non-Euclidean Geometry.

Professor H. A. Rowland, one of America's most noted physicists, died at Baltimore April 16. Professor Rowland was for a number of years past, Director of the Physical Laboratory at Johns Hopkins University.

Dr. E. A. Engler, Professor of Mathematics in Washington University St. Louis, Mo., has been elected President of the Worcester Polytechnic Institute to succeed Dr. T. C. Mendenhall who resigned on account of ill health.

Dr. G. A. Miller has recently been appointed Secretary of Section A of the American Association for the Advancement of Science in place of Professor Lord who resigned. Papers for the program of the Denver meeting should therefore be sent to Dr. Miller, 115 Cook St., Ithaca, N. Y.

BOOKS AND PERIODICALS.

Elements of Astronomy. By Simon Newcomb, Ph. D., LL. D., late Professor of Mathematics and Astronomy, Johns Hopkins University, formerly Senior Professor of Mathematics, U. S. Navy, and Superintendent of the American Ephemeries and Nautical Almanac. 1877-97. 12mo, cloth, 240 pages. Price, $1.00. New York and Chicago: American Book Co.

The general scope and plan of the book may be seen from the titles of the chapters: I. Relation of the Earth to the Heavens; II. The Revolution of the Earth Round the Sun; III. of Time; IV. Observation and Measurement of the Heavens; V. Gravitation; VI. The Earth; VII. The Sun. This book is written by America's foremost astronomer and, therefore, so far as the matter of the work is concerned no criticism can be offered. Neither can any criticism be offered as to the method of presentation. The author has certainly succeed in making a text-book simple and lucid enough to be comprehended by any one who has mastered the elements of arithmetic and the most rudimental principles of geometry, and at the same time it is full and complete enough for more advanced readers.

B. F. F.

The Riddle of the Universe at the Close of the Nineteenth Century. By Ernst Haeckel, Ph. D., M. D., LL. D., Sc. D., and Professor at the University of Jena, author of "The History of Creation," "The Evolution of Man," etc., translated by Joseph McCabe. 8vo, cloth, 390 pages. Price, $1.50. New York and London: Harper & Bros.

The scope of the work may be inferred from the titles of the chapters: I. The Nature of the problem; II. Our Bodily Frame; III. Our Life; IV. Our Embryonic Develop-

ment; V. The History of Our Species; VI. The Nature of the Soul; VII. Psychic Gradations; VIII. The Embryology of the Soul; IX. The Philogony of the Soul; X. Consciousness; XI. The Immortality of the Soul; XII. The Law of Substance; XIII. The Evolution of the World; XIV. The Unity of Nature; XV. God and World; XVI. Knowledge ahd Belief; XVII. Science and Christianity; XVIII. Our Monistic Religion; XIX. Our Monistic Ethics; XX. Solution of the World's Problems; Conclusion.

While the author has not solved the great "Riddle" to the entire satisfaction of every one who reads this work, yet he has handled the difficult problem with which he has dealt in a masterly way. A solution of this problem to satisfy all shades of belief is impossible, for there is only one solution, but there is an infinitude of beliefs and opinions. To read the correct solution, should it ever be given, one must lay aside all shades of beliefs and opinions and follow with an unbiased mind, step by step, the argument set forth. The author has not, in the light of more recent scientific discovery and progress, seen fit to change his opinions in reference to these great problems as formulated and expressed a quarter of a century ago. B. F. F.

The Measurement of General Exchange-Value. By Correa Moylan Walsh. Large 8vo, cloth, 580 pages. Price, $3.00, New York and London: The Macmillan Co.

This work should be in the hands of every teacher of Political Economy. In dealing with his subject, the author does not use any more mathematics than may be fairly expected to be known by a liberally educated person. B. F. F.

A Short History of the Greeks from the Earliest Times to B. C. 146. By Evelyn S. Shuckburgh, M. A., Late Fellow of Emanuel College, Cambridge. Author of "A Translation of Polybius," etc. 8vo. Cloth, 388 pages. Price, $1.10. New York and London: The Macmillan Co.

In this book, the story of the early Greeks is told with unusual interest. In developing the story, the author has chosen those topics which illustrate the political life and intellectual activities of the Greeks wherever they lived. B. F. F.

American Journal of Mathematics. Edited by Frank Morley, with the cooperation of Simon Newcomb, A. Cohen, Charlotte A. Scott, and other mathematicians. Published under the auspices of Johns Hopkins University. Issued quarterly. Price, $5.00 per year. Single Number, $1.50.

Number 2, Vol. XXII, contains the following articles: The Cross-ratio Group of 120 Quadratic Cremona Transformation of the Plane, by H. E. Slaught; Memoir on the Algebra of Symbolic Logic, by A. N. Whitehead; On a Special Form of Annular Surfaces, by Virgil Snyder; On the Transitive Substitution Groups whose Order is a Power of a Prime Number, by G. A. Miller; Geometry on the Cubic Scroll of the Second Kind, by Frederick C. Ferry. B. F. F.

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BIOGRAPHY.

CHARLES HERMITE.

By DR. GEORGE BRUCE HALSTED, Austin. Texas.

The fourteenth of Janunry, 1901, should be marked with a black stone in the annals of mathematics. Then the eminent geometer, the incomparable man, the great Hermite, one of the glories most pure of France, was lost to science, and implacable death threw into mourning his family, his friends, and his admirers.

As mathematician of the first rank he leaves to the glory of his country and of all humanity a superb scientific monument erected in sixty years completely dedicated to "his dear *analyse*" (to use one of his phrases) and to preparing by the infusion of his genius placed at the service of teaching that galaxy of illustrious mathematicians who now so much adorn our sister nation. Like Sturm, he united in an extraordinary degree the qualities of professor who wins the love of his disciples to those of one who inculcates the love of science for science. Endowed, like his compatriots, Pascal and Clairaut, with singular precocity, we see him, yet a scholar of the Lyceum Louis le Grand, win the prize for mathematics with a noteworthy thesis, and shortly after, as student of the Polytechnic School, attract the attention of Jacobi with his first works and place himself as of right in the first rank among the analysts of Europe.

It is not our object to make a minute analysis of the works of the great

geometer, to which would be necessary time and competence that we lack: our aim is much more modest: we seek to render what is heart-felt homage to the man we have so deeply venerated and from whom we have received infinite proofs of benevolence during the fifteen or sixteen years that we have had the honor to possess his friendship in so many ways precious. It is not possible, speaking of Charles Hermite, to fail to say how in the higher analysis, in algebra and in the theory of numbers one encounters everywhere the footprints of his giant tread. How could we leave unmentioned his memoir on the exponential function, where in demonstrating the transcendence of the number e he opens the way which eleven years after conducted Lindemann to the demonstration of the analogous property of π, solving in negative form the celebrated problem which for two thousand years had in vain fatigued geometers?

Nor can we pass in silence the enormous contribution which Hermite brought to the *Theory of Forms:* his law of reciprocity, his admirable researches on associate covariants, his works on quintic forms, his memoir on the equation of the fifth degree, and his celebrated theorem having Sturm's as corallary.

The works of Charles Hermite in the theory of functions are a new revelation of his genius. His profound investigations on Abelian functions, their division and their transformations, as also those relative to elliptic functions, form a monument of glory erected to French science, disclosing the sagacity of the grand analyst in the facility with which are deduced from the most lofty analytic investigations, corollaries which unveil difficult properties of the theory of numbers.

Neither can we neglect to mention the work "*Sur quelques applications des fonctions elliptiques*" (1885), of which only the first part was published: in this are found the beautiful applications of these functions which conduct him to the general integral of the equation of Lamé on the equilibrium of temperature of a homogeneous ellipsoid, which leads the author, in two particular cases, to the study of the rotation of a solid body around a fixed point (when there do not exist accelerating forces), treated by Jacobi, and to the consideration of the conic pendulum. So far as we know, Hermite leaves two didactic works: his "*Cours de la faculté des sciences de Paris*" (1891), and his "*Note sur la théorie des fonctions elliptiques*" (168 pages) which serves as appendix to the *Cours de calcul differentiel et integral* of J. A. Serret (4th ed. 1894). We have from him also two brief but interesting notes on the invariants of binary forms of the 5th and 6th order in the French translation of Salmons Higher Algebra.

The French geometer had the good fortune not granted all great men to see recognized in his lifetime by the scientific world his extraordinary merit. The 24th of December, 1892, his sixtieth birthday, the friends, the disciples, the admirers of the great geometer assembled at the Sorbonne to present him the gold medal struck in his honor by international subscription. The illustrious artist, Chaplain, cut upon it the bust of the one commemorated and translated onto metal with admirable fidelity his venerable face, affable and frank, illuminated by the scintilla of genius. The Minister of Public Instruction, M. Ch.

Dupuy, presented to Hermite in the name of the President of the Republic the insignia of Grand Officer of the Legion of Honor, and the messages were read of those who from various parts of the world associated themselves with the splendid ceremony.

High testimony of admiration and sympathy was offered the great geometer more recently upon the occasion of the meeting at Paris, last August, of the international congress of mathematicians. The Congress sent him a telegram of admiration and sympathy (he was at Saint-Jean-de Luz). This act caused vast satisfaction and profound emotion to the scientist, as he wrote me in one of his last letters.

Hermite retained to the last day of his life his privileged intelligence ; but his body suffered. In a long letter of his, a few days before his death, he complained of his attacks of asthma and of the lack of appetite and of sleep : he seemed to forsee the nearness of his end, so that sending me one of his works he said that this would be without doubt *the last* ! and that he had in great part accomplished it at Saint-Jean-de Luz, whereby the benefit of the mild climate had reawakened his mathematical activity. This last work is a letter to Professor Pincherle published in tomo V of the "*Annali di Matematica.*" He told us also that he had sent a brief article to the new journal "Le Matematiche" of Prof. Alasia.

We will end by expressing a wish. We wish that those who have the authority would take the initiative toward an international subscription for a work containing an extended biography of the ever memorable geometer, and a minute analysis of his works ; perhaps might be added some brief articles by very illustrious living mathematicians ; something, in fine, which would be as a funeral crown offered to the memory of the great dead.

[Written by Juan J. Durán-Loriga for *Le Matematiche*, and translated by the English editor, G. B. Halsted.]

A PROBLEM AND ITS SOLUTION.

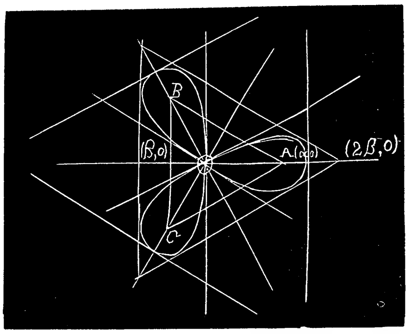
By EUPLIO CONOSCENTE, B. Sc., Math. D., Member of Circolo Mathematico di Palermo, New York City.

Find that one of these curves $(x^2+y^2)^3=a^3(x^3-3xy^2)$.

I. (a) is the locus of such points that the product of their distances from the vertices of a fixed equilateral triangle is equal to the semi-parameter. (b) The bitangents each touching the curve in two real distinct points are parallel to the sides of the fixed triangle and their six points of contact are on a circle. (c) The sides of the equilateral triangle obtained by the tangents each touching the curve in its three farthest real points from the origin of coördinates are parallel to the tangents of its real triple points. (d) Some other property showing the form of this curve.

II. (a) The "deficiency" (genus) of the curve is $p=1$ and it has three triple points and their tangents are all distinct. (b) Its arc is expressed by the Abelian integral of the first kind attached to its equation (saving a constant). (c) x and y are then two elliptic functions of this arc.

SOLUTION.



I. (a) Let the origin of rectangular coordinates, O, be the center of an equilateral triangle; one of its sides, BC, parallel to y-axis; α, being the arithmetical value of $\frac{\sqrt[3]{a^3}}{2}$ the radius of its circumscribed circle $x^2+y^2=\alpha^2$. Then its vertices are $(x=\alpha,\ y=0)$, $(x=-\frac{1}{2}\alpha,\ y=\frac{\sqrt{3}}{2}\alpha)$, $(x=-\frac{1}{2}\alpha, y=\frac{\sqrt{3}}{2}\alpha)$ (1).

The locus of such points that the products of their distances from the points (1) is $\frac{1}{2}a^3$ is expressed by the equation

$$\sqrt{\{[(x-\alpha)^2+y^2][(x+\tfrac{1}{2}\alpha)^2+(y-\tfrac{\sqrt{3}}{2}\alpha)^2][(x+\tfrac{1}{2}\alpha)^2+(y+\tfrac{\sqrt{3}}{2}\alpha)^2]\}}=\alpha^3.$$

Developing and reducing this equation we get

$$(x^2+y^2)^3=\alpha^3(x^3-3xy^2)\ldots.(2).$$

The sides of the fixed triangle are

$$x+\alpha=0,\quad x-\sqrt{3}y-2\alpha=0,\quad x+\sqrt{3}y-2\alpha=0,\quad (\alpha^3=\tfrac{1}{2}a^3)\ldots.(3).$$

(b) These three straight lines

$$x+\beta=0,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y-2\beta=0,\quad (\beta^3=\tfrac{1}{4}\alpha^3)\ldots(4),$$

are bitangent to the curve respectively each in one of the following three couples of points:

$$(x=-\beta,\ y=\pm\beta),$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta,\ y=-\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=-\frac{\sqrt{3}-1}{2}\beta)..(5).$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta,\ y=\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=\frac{\sqrt{3}-1}{2}\beta).$$

The bitangents (4) intersect each other in three points which form the vertices of an equilateral triangle:

$$(x=2\beta,\ y=0),\quad (x=-\beta,\ y=-\sqrt{3}\beta),\quad (x=-\beta,\ y=\sqrt{3}\beta)\ldots.(6),$$

and it is evident that the straights (3) and (4) are three couples of parallels.

All six points (5) stay on the circle $x^2+y^2=2\beta^2$.

(*c*) The circle $x^2+y^2=a^2$ *touches* the curve in its real farthest points from the origin of coördinates

$$(x=a,\ y=0),\quad (x=-\tfrac{1}{2}a,\ y=\frac{\sqrt{3}}{2}a),\quad (x=-\tfrac{1}{2}a,\ y=-\frac{\sqrt{3}}{2}a)\ldots.(7).$$

Indeed these points are the limits of three couples of real points obtained through intersection with a circle, center O and radius r, variable from 0 to a, for if $r>a$, all twelve points of intersection would be imaginary.

The tangents which touch both this circle and curve are

$$x-a=0,\quad x-\sqrt{3}y+2a=0,\quad x+\sqrt{3}y+2a=0\ldots\ (8).$$

The origin of coördinates is a real triple point of the curve (we will prove later it is alone) for the first and second partial derivatives of (2) vanish for $x=0$, $y=0$; its tangents are three distinct ones

$$x=0,\quad x-\sqrt{3}y=0,\quad x+\sqrt{3}y=0\ldots.(9),$$

which are parallel to the straights (8).

(*d*) The straight lines (3), (4), (8), and (9) give three sets of four parallel and their three directions are inclined 90°, 60°, and 30° to x-axis. The x-axis is an axis of symmetry of the curve for the degree of y in (2) is always a multiple of 2 ; on this straight lie the first points of (1), (6), and (7). Through two successive rotations (120°) of the coördinate system we get that the straights containing respectively the second and third points (1), (6), and (7) are also axes of symmetry and there is no other one. Then we can say that the curve is a three-lobed one, each branch has finished values and one of them is within the following straights, *i. e.*:

$$x-\sqrt{3}y=0,\quad x+\sqrt{3}y-2\beta=0,\quad x=a,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y=0,$$

by their points of contact and some other one we can get, it is easy to obtain the form of the curve.

II. (*a*) The curve $x'^3-3x'y'^2=1/a^3$ is the transformed one of (2) through reciprocal vectorial rays and its genus or deficiency is $p=1$; then also the first curve's genus is $p=1$, this transformation being a birational one. The curve (2) has three triple points : the origin of coördinates (a real one) and the two cyclic points [(1, i, 0) and (1, $-i$, 0)] and no more multiple points for $p=1$. The tangents of the real triple points are expressed by the equations (8) and those of the imaginary points are $(x\pm iy)^3=\frac{1}{2}a^3\ z^3$, which are six distinct tangents, of superior inflexion, for each branch is linear and of the third class.

(b) The equation of the curves (2), if written in polar coördinates, is

$$\rho^3 = a^3 \cos 3\theta \ldots . (10).$$

The differential expression of its arc s is

$$ds = \frac{a^3 d\theta}{\rho^2} \ldots . (11),$$

or, in Cartesian coördinates,

$$ds = \frac{a^3}{x^2+y^2} d \text{ arc tg} \frac{y}{x},$$

which is a rational function of x and y, and then $\int ds$ is an Abelian integral attached to the equation (2).

From (10) we get $\frac{d\theta}{\rho^2} = \frac{d\rho}{a^3 \sin 3\theta}$, then $s = \int \frac{a^3 d\rho}{\sqrt{(a^6 - \rho^6)}}$, which has a finished value anywhere and thus it is an Abelian integral of the first kind. But the genus of the curve is $p=1$ and it must have only an integral of the first kind attached to its equation, and then, saving a constant, this integral expresses its arc s.

Let us put $\rho^2 = 1/r$; we have

$$s = \int \frac{dr}{\sqrt{[4r^3 - (4/a^6)]}} \ldots (12),$$

which is an elliptic integral of the first kind.

The well known fundamental relation of Weierstrass between his function pu and its first derivative $p'u$, $p'^2u = 4p^3u - g_2 pu - g_3$, will be here

$$p'^2 s = 4p^3 s - 4/a^6 \ldots . (13).$$

The coördinates x and y of the curve (2) are expressed by the following functions

$$x = -\frac{3}{a^3} \frac{ps}{p^3 s - (4/a^6)}, \quad y = \frac{\sqrt{-3}}{2} \frac{psp's}{p^3 s - (4/a^6)} \ldots . (14),$$

which are two elliptic functions of the arc s. Indeed eliminating ps and $p's$ between (13) and (14) we get

$$(x^2+y^2)^3 = a^3(x^3 - 3xy^2) = \left(\frac{-3p^2 s}{p^3 s - (4/a^6)} \right)^3.$$

It ought to be so for the coördinates of an elliptic curve are expressed by elliptic functions of the Abelian integral of the first kind relative to its equation.

ON THE CONCEPTS OF NUMBER AND GROUP.

By DR. G. A. MILLER, Cornell University, Ithica, N. Y.

There is a marked difference between the methods generally employed in the teaching of elementary geometry and arithmetic. In the former it is customary to begin by assuming that all the elements—points, lines, and planes—exist. By means of a sufficient number of postulates (axioms) these elements are then combined in such a way as to lead to a large number of interesting and valuable consequences. To complete the theory it is only necessary to prove that the postulates do not lead to any contradictory results and that they are sufficient to prove all geometric theorems.

In the theory of arithmetic we generally employ a widely different method. To develop the fundamental concept (number) we begin with the positive integers. This concept is gradually extended by the introduction of fractional, negative, irrational, and complex numbers. The last class may be regarded as pairs of real numbers. This method has been called the *genetic* method* because the general concept of number is reached by the gradual extension of the more elementary concept. The method described in the preceding paragraph may be distinguished as the *axiomatic* method.

While the genetic method seems to be the most suitable for the teaching of the elements of arithmetic it is doubtful whether it is so well suited for a logical presentation of the theory of this subject. At any rate it is a matter of great interest to develop the theory of arithmetic by the axiomatic method. The axioms of arithmetic have recently received considerable attention. The works of Dedekind, Burali-Forti, Padoa, Pieri, Peano, and others, along this line have lead to many interesting results. The last volume of th German Mathematical Association contains a paper by Professor Hilbert in which the concept of real numbers is presented according to the axiomatic method as follows :†

There is a system of elements (things) which are called numbers and they are represented by a, b, c, ... Their relations to each other and their laws of combination are completely defined by the following axioms :

A. Axioms of Combination.

1. By adding a and b we obtain a definite number c, in symbols :

$$a+b=c \text{ or } c=a+b$$

2. If a and b are given there is always one and only one number x and also one and only one number y such that

$$a+x=b, \quad y+a=b$$

*Hilbert, Jahresbericht der Deutschen Mathematik Vereinigung, 1900, page 180.
†Loc. cit.

3. There is a definite number, viz. 0, such that every number a satisfies the equations

$$a+0=a \text{ and } 0+a=a$$

4. According to a second law of combination, called multiplication, we obtain a definite number c from a and b, in symbols

$$ab=c \text{ or } c=ab$$

5. When a and b are any given numbers, except a cannot be 0, there is one and only one number x and also one and only one number y such that

$$ax=b, \ ya=b$$

6. There is a definite number, viz. 1, such that every number a satisfies the equations

$$a.1=a \text{ and } 1.a=a$$

B. Axioms of Operation.

The following formulas apply to any three numbers; a, b, c.

1. $a+(b+c)=(a+b)+c$.
2. $a+b=b+a$.
3. $a(bc)=(ab)c$.
4. $a(b+c)=ab+ac$.
5. $(a+b)c=ac+bc$.
6. $ab=ba$.

C. Axioms of Arrangement.

1. If a and b are two different numbers then one (say a) of them must be larger than the other (b). In this case b is said to be smaller than a, in symbols $a>b$ and $b<a$.

2. From $a>b$ and $b>c$ it follows that $a>c$.

3. When $a>b$ the following inequalities must hold $a+c>b+c$ and $c+a>c+b$.

4. From $a>b$ and $c>0$ it follows that $ac>bc$ and $ca>cb$.

D. Axioms of Continuity.

1. If a and b are any two numbers such that $a>0$ and $b>0$ it is always possible to add a to itself a sufficient number of times so that the sum is greater than b; *i. e.*

$$a+a+\ldots.+a>b$$

2. The numbers a, b, c, ... form a system which is completely defined by the preceding axioms ; *i. e.* it is impossible to find a different system of elements which obey all of these conditions.

Professor Hilbert observed that some of these axioms can be deduced from the others—in particular, it is easy to prove that the existence of 0 is a consequence of A 1, 2 and B 1, 2. From A 1, 2 it follows that for every number a there is a number x such that

$$a+x=a \text{ or } b+(a+x)=b+a$$

and that $a+b$ may represent any number while a is fixed. From B 1 we observe that $b+(a+x)=(b+a)+x$; *i. e.* this number x is independent of a. Finally, we conclude from B 2 that this number x is the same in

$$a+x=a \text{ and } x+a=a$$

This proves that axiom A 3 is a consequence of A 1, 2 and B 1, 2.

Most of the thoughts expressed above may be found in the article by Professor Hilbert to which reference has been given. As the thoughts seem likely to interest teachers of mathematics and as the journal in which they appeared is probably inaccessible to most of the readers of this Journal it seemed desirable to reproduce them here with slight modifications. In his address before the Paris International Mathematical Congress, Professor Hilbert stated that one of the interesting mathematical problems which await solution is the determination of some one system of independent compatible axioms governing and defining arithmetic concepts. In the discussion of this paper Peano of Italy stated that several Italian mathematicians, viz. Burali-Forti, Padoa, and Pieri, had completely solved this problem in memoirs to which references are given in Vol. 7, of Rivista di Matematica.

It is interesting to observe that all the axioms of the group theory are included in the above axioms of arithmetic. The system of axioms (completely defining and governing the concepts of group theory) which is frequently given, is composed of A 4, 5 and B 3. When the order of the group is finite A 5 may be replaced by the following simpler axiom : From each of the equations

$$ab=ab' \text{ or } ba=b'a$$

it results that $b=b'$. It is thus apparent that the concept of operator in the theory of groups is much less restricted than the concept of real numbers, and includes the latter with respect to multiplication. Numbers have, however, an additional law of combination, viz. addition, while the operators of a group are combined according to a single law of combination generally called multiplication.

LUCUS A NON LUCENDO.

By A. LATHAM BAKER, Ph. D., University of Rochester, Rochester, N. Y.

I reproduce here a curious piece of mathematical legerdemain which proves without proving. It has been, with its variations, the standard for many years in most all the text books. I think it will be difficult to find a proof (?) more utterly lacking in every pedagogical requirement, and yet correct in its final conclusion.

THEOREM. *Any continuous arc of the evolute is equal to the difference between the radii of curvature of the involute that are tangent to the arc at its extremities.*

If r is the radius of curvature and a, b the coördinates of the center of curvature, then the circle of curvature is

$$(x-a)^2+(y-b)^2=r^2 \ldots.(1).$$

But for a normal through a, b we get

$$y-b=-\frac{dx}{dy}(x-a)\ldots.(2),$$

and since the normal is tangent to the evolute

$$y-b=\frac{db}{da}(x-a)\ldots.(3).$$

From (1) and (2), if we suppose the point x, y to move along the curve, and therefore y, a, b, and r to be functions of x, we get

$$(x-a)dx+(y-b)dy-(x-a)da-(y-b)db=rdr,$$

and from (2), $(x-a)dx+(y-b)dy=0$, whence

$$(x-a)da+(y-b)db=rdr\ldots.(4).$$

From (3) and (4), $(x-a)\dfrac{da^2+db^2}{da^2}=-rdr\ldots.(5).$

From (1) and (3), $(x-a)^2\dfrac{da^2+db^2}{da^2}=r^2\ldots.(6).$

(5) squared, divided by (6), gives $da^2+db^2=dr^2=ds^2$.

Whence $ds=\pm dr\ldots.(7)$. Q. E. D.

Now, to begin with, calling r the radius of curvature does not make it such. We might just as rightfully call it *not* the radius of curvature. Under this latter hypothesis what does equation (7) prove?

To call such a sequence of *disjecta membra* a proof is little short of absurdity, however correct the formal algebraic operations may be. One seldom runs across such a total lack of syllogistic sequence of thought presented under the guise of proof.

Let us see what the r of equation (7) really does represent.

(1) puts a, b at the center of a circle through x, y.

(2) puts *another* a, b on the normal through the locus of another x, y.

(3) puts a third x, y on the tangent to the locus of a, b.

(4)=1st condition+2d condition. Unites the first and second x, y, and the first and second a, b.

(5)=4th condition+3d condition, puts a, b on the normal to x, y, and at the center of circle r, and *tangent to the normal*, and *therefore* on the evolute, at the center of curvature, and the arbitrary r now becomes the definite R, the radius of curvature, and the a, b becomes α, β, the coördinates of the center of curvature.

(6)=1st condition+3d condition, puts a, b at center of circle through x, y and tangent to the *secant* through x, y, and leaves r and a, b arbitrary.

(7)=5th condition+6th condition, converts the r of (6) into the R of (5), and the a, b into α, β, and hence (7) becomes $\surd(d\alpha^2+d\beta^2)=dS=\pm dR$, and the theorem follows.

It will be noticed that there is not a suspicion of anything pertaining to the evolute until we reach equation (5), and that then by conjunction of conditions the thing snaps into place and the a, b locus becomes, *willy nilly*, the α, β locus or evolute, and the r similarly becomes R, the radius of curvature. The terminology applied to a, b, r is ineffective. It is the conjunction of conditions that decide what they should be called.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

142. Proposed by **M. J. CRAWFORD**, Principal of Crawford's Academy, Savannah, Ga.

A gentleman has a garden 400 feet long and 300 feet wide, which he wishes to raise 9 inches higher by means of the earth to be dug out of a ditch 6 feet wide and surrounding the entire garden. How deep must the ditch be?

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$400\times300\times\frac{9}{12}=90000$ cubic feet of earth necessary to raise the garden.

$2(412+300)\times6=8544$ square feet, area of surface of ditch.

$90000\div8544=10\frac{95}{178}$ feet, depth of ditch.

143. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A's income $=a/b$th part $=\frac{3}{4}$ of B's income. A's outgo $=m/n$th part $=\frac{1}{2}$ of B's income. B's outgo $=p/q$th part $=1/1$ of A's income. What is the ratio of their savings?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A's outgo $=m/n$th of B's income.

A's saving $=\left(\frac{a}{b}-\frac{m}{n}\right)=\frac{an-bm}{bn}$ of B's income.

B's outgo $=ap/bq$th of B's income.

B's saving $=1-\frac{ap}{bq}=\frac{bq-ap}{bq}$ of his income.

$$\therefore \frac{an-bm}{bn}:\frac{bq-ap}{bq}=\frac{an-bm}{n}:\frac{bq-ap}{q},$$

but $a=3$, $b=4$, $m=p=q=1$, $n=2$.

$\therefore$ The ratio of their savings is 1:1.

ALGEBRA.

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117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Rationalize $l^{\frac{1}{2}}+m^{\frac{1}{2}}+n^{\frac{1}{2}}+x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0$.

Solution by the PROPOSER.

Let $u^6+pu^4+qu^3+ru^2+su+t=0$, be the equation having $l^{\frac{1}{2}}$, $m^{\frac{1}{2}}$, $n^{\frac{1}{2}}$, $x^{\frac{1}{2}}$, $y^{\frac{1}{2}}$, $z^{\frac{1}{2}}$ for its roots; then $u^3+pu^2+qu^{\frac{3}{2}}+ru+su^{\frac{1}{2}}+t=0$ is the equation having l, m, n, x, y, z for its roots.

$$\therefore u^3+pu^2+ru+t+u^{\frac{1}{2}}(qu+s)=0$$

$$\text{or } u^6+2pu^5+(p^2+3r)u^4+(2t+2pr-q^2)u^3 +(r^2+2pt-2qs)u^2+(2rt-s^2)u+t^2=0.$$

Let $a=l+m+n+x+y+z=\Sigma(l)$.

$b=lm+ln+lx+ly+lz+mn+mx+my+mz+nx+ny+nz+xy+xz+yz=\Sigma(lm)$.

$c=lmn=lmx+lmy+lmz+lnx+lny+lnz+lxy+lxz+lyz+mnx+mny+mnz+mxy+mxz+myz+nxy+nxz+nyz+xyz=(\Sigma(lmn)$.

$d=lmnx+lmny+lmnz+lnxy+lnxz+lxyz+lmxy+lmxz+lmyz+lnyz+lxyz+lmxy+lmxz+lmyz+mnxy+mnxz+mnyz+mxyz+nxyz=\Sigma(lmnx)$.

$e=lmnxy+lmnxz+lmnyz+lnxyz+lmxyz+mnxyz=\Sigma(lmnxy)$.

$f=lmnxyz$.

Then $a=2p$, $b=p^2+2r$, $c=2t+2pr-q^2$, $d=r^2+2pt-2qs$, $e=2rt-s^2$, $f=t^2$.

$\therefore p=\frac{1}{2}a$, $r=\frac{1}{8}(4b-a^2)$, $q^2=\frac{1}{8}(16t+4ab-a^3-8c)$..(1), $s^2=\frac{1}{4}(4bt-a^2t-4e)$..(2).

$qs=(16b^2-8a^2b+a^4+64at-64d)/128$....(3).

(1)$\times$(2)$=[(3)]^2$.

$\therefore$ $[(16b^2-8a^2b+a^4-64d)/128]^2+(at/128)(16b^2-8a^2b+a^4-64d)$
$+\frac{1}{4}a^2t^2=(64bt^2-16a^2t^2-16abe+4a^3e+32ce)/32+(16ab^2-4a^3b-32bc-4a^3b+a^5+8a^2ct-64e)t/32$.

But $f=t^2$.

$\therefore$ $[(16b^2-8a^2b+a^4-64d)/128]^2+(96a^2f-256bf+64abe-16a^3e-128ce)/128=(48ab^2-24a^3b-256e-128bc+3a^5+32a^2c+64ad)t/128$.

$\therefore\{(16b^2-8a^2b+a^4-64d)^2+[128(96a^2f-256bf+64abe-16a^3e-128ce)]\}^2=[128(48ab^2-24a^3b+3a^5+32a^2c+64ad-128bc-256e)]^2f$.

The values of a, b, c, d, e, f in this last equation is the rationalized form in full.

Also solved by *ELMER SCHUYLER*.

118. Proposed by FREDERIC R. HONEY, Ph. B., Instructor at Trinity College, and Lecturer at Smith College, New Haven, Conn.

An army whose length is equal to a, moves forward. An officer is sent from the rear to the van, and is required to present himself at the rear again when the rear has reached the point where the van was when the army began to move. How far did the officer travel?

I. Solution by H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.; S. F. NORRIS, Baltimore City College. Baltimore, Md.; C. E. ARMENTROUT, Rockingham Military Institute, Mt. Crawford, Va.; J. W. YOUNG, Cornell University, Ithaca, N. Y.; M. E. GRABER, Heidelberg University, Tiffin, Ohio; P. S. BERG, Larimore. N. D.; J. F. TRAVIS, Student Ohio State University, Columbus, O.; MARTIN J. SPINKS, Wilmington, O.; and the PROPOSER.

When the officer moves forward $a+x$, the army goes x; when the officer moves backward x, the army goes $a-x$.

Therefore, $\dfrac{a+x}{x}=\dfrac{x}{a-x}$, whence $x=\dfrac{a}{\sqrt{2}}$.

Whole distance traveled by the officer$=a+2x=a(1+\sqrt{2})$.

II. Solution by C. HORNUNG. A. M., Heidelberg University, Tiffin, O.; D. G. DORRANCE, Jr., Camden, N. Y.; JOSIAH H. DRUMMOND, LL. D.. Portland, Me.; and the late SYLVESTER ROBINS, North Branch, N. J.

Let x=officer's rate, and y the army's rate of travel.

Then $\dfrac{a}{x-y}+\dfrac{a}{x+y}=\dfrac{a}{y}$. Whence $x=y(1\pm\sqrt{2})$, or $x/y=1+\sqrt{2}=2.414+$

that is, the officer travels 2.414 times as fast as the army, and, therefore, 2.414 times as far. Therefore the officer travels $(1+\sqrt{2})a$.

Also solved by *J. SCHEFFER*, *COOPER D. SCHMITT*, *H. C. WILKES*, and *G. B. M. ZERR*.

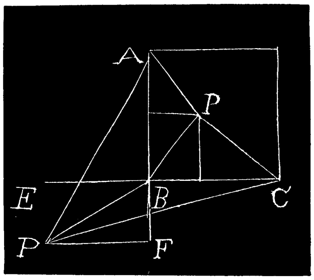
GEOMETRY.

151. Proposed by FRANK A. GRIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

A point is at a distance of 1 inch, 2 inches, and $2\frac{1}{2}$ inches, respectively, from three corners of a square. Construct the square. Also solve for the general distances a, b, c.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; P. S. BERG, B. Sc., Larimore, N. D.; G. W. DROKE, State University, Lafayette, Ark.; and MARTIN SPINKS, Wilmington, Ohio.

Let P be the point, $PA=a$, $PB=b$, $PC=c$, $AB=x$=side of square, $PE=m$, $PF=n$, $m^2+n^2=b^2$....(1).



$a^2-(x+m)^2=a^2-x^2-2mx-m^2=n^2$....(2).

$c^2-(x+n)^2=c^2-x^2-2nx-n^2=m^2$....(3).

Adding (2) and (3) and using (1) we get

$$n+m=\frac{a^2+c^2-2b^2-2x^2}{2x}\ldots.(4).$$

Subtracting (3) from (2) we get

$$n-m=\frac{c^2-a^2}{2x}\ldots(5).\quad \therefore n=\frac{c^2-b^2-x^2}{2x},\quad m=\frac{a^2-b^2-x^2}{2x}.$$

These values of m and n in (1) give

$$x^4-(a^2+c^2)x^2=b^2(a^2+c^2)-\frac{a^4+c^4+2b^4}{2}.$$

$$\therefore x=\{\tfrac{1}{2}[a^2+c^2\pm\sqrt{4b^2(a^2+c^2-b^2)-(a^2-c^2)^2}]\}^{\frac{1}{2}}$$

$=1.5163$ inches when P is without the square.

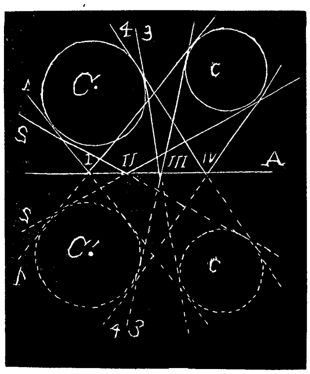
$=2.8197$ inches when P is within the square.

I. When P is without, lay off $PE=m$, perpendicular to PE lay off $EB=n$ and $BC=x$, also AB perpendicular to $EB=x$. The square is now determined.

II. Similarly, when P is within, only $EC=x-n$.

152. Proposed by ELMER SCHUYLER, Reading, Pa.

Find a point in a given straight line such than tangents drawn from it to two given circles shall make equal angles with the line.



Solution by MARCUS BAKER, Washington, D. C.

In the annexed figure let A be the given line, and C and c the given circles.

Regarding line A as an axis, revolve C and c about it until they fall at C' and c'.

Then draw the four common tangents 1, 2, 3, and 4 of circles C and c' intersecting the axis in I, II, III, and IV. Each of these points fulfils the conditions. The proof is obvious.

Also solved by *G. B. M. ZERR*, and *D. B. NORTHROP*.

CALCULUS.

110. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the volume removed by boring an auger hole through a right circular cone, the radius of the auger being r, the radius of the cone R, and the altitude h, and the axis of the auger intersecting axis of the cone at right angles and at a distance c from the vertex.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $R^2/h^2=a^2$, and suppose the cone to have one nappe.

Then $y^2+z^2=a^2x^2$ is the equation to the cone, and $(x-c)^2+y^2=r^2$ is the equation to the auger.

When $y=ax$, $z=0$, $x_1=\dfrac{c+\sqrt{[r^2(a^2+1)-a^2c^2]}}{a^2+1}$, $x_2=\dfrac{c-\sqrt{[r^2(a^2+1)-a^2c^2]}}{a^2+1}$

Let $\sqrt{[r^2-(x-c)^2]}=y'$, $[r^2-(x-c)^2][a^2x^2-r^2+(x-c)]=A^2$.

I. When $c<r$.

$$V=4\int_{x_1}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}dxdy+4\int_0^{x_1}\int_0^{ax}\sqrt{[a^2x^2-y^2]}dxdy$$

$$=2\int_{x_1}^{r+c}\left(A+a^2x^2\sin^{-1}\frac{y'}{ax}\right)dx+\pi a^2\int_0^{x_1}x^2dx$$

$$=2\int_{x_1}^{r+c}\left(A+\frac{a^2x^2(cx+r^2-c^2)}{3A}\right)dx+\tfrac{1}{3}\pi a^2x_1^3=2\int_{x_1}^{r+c}Bdx+\tfrac{1}{3}\pi a^2x_1^3.$$

II. $c>r$, but $r<ac/(a^2+1)$.

$$V=4\int_{c-r}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}dxdy=2\int_{c-r}^{c+r}Bdx.$$

III. $c>r$, but $r>ac/(a^2+1)$.

$$V=4\int_{x_1}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}dxdy+4\int_{c-r}^{x_2}\int_0^{y'}\sqrt{[a^2x^2-y^2]}dxdy$$

$$+4\int_{x_2}^{x_1}\int_0^{ax}\sqrt{[a^2x^2-y^2]}dxdy=2\int_{x_1}^{c+r}Bdx+2\int_{c-r}^{x_2}Bdx+\tfrac{1}{3}\pi a^2(x_1^3-x_2^3).$$

Let $u=x-c$ and $l^2=R^2+h^2$.

$$\therefore A=\frac{1}{hl}\sqrt{\{(r^2-u^2)[(l^2u+R^2c)^2+h^2(R^2c^2-l^2r^2)]\}}.$$

Let $h^2(R^2c^2-l^2r^2)=b^2l^4$, $R^2c/l^2=d$.

Then $A=(l/h)\surd\{(r^2-u^2)[b^2+(u+d)^2]\}$.

This can be integrated by a process similar to that employed by Professor Finkel in Vol. V, No. 1, pages 20, 21.

Let $Rc-lr$, then $A=\frac{1}{h}(lu+Rr)\surd(r^2-u^2)=\frac{R}{rh}(cu+r^2)\surd(r^2-u^2)$.

$$\therefore\ V=\frac{2R}{hr}\int_{-r}^{r}(cu+r^2)\surd(r^2-u^2)du+\frac{2\,Rr}{3h}\int_{-r}^{r}\frac{(u+c)^2du}{\surd(r^2-u^2)}$$

$$=\frac{2\pi\,Rr}{3h}(2r^2+c^2).$$

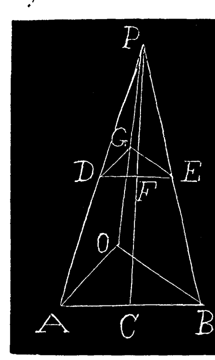
111. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

(a). Find the dimensions of a cup, capacity c, in the form of a frustum of a regular pyramid of n faces, so that its internal surface is a minimum.

(b). Find the dimensions of a cup, capacity c, in the form of a frustum of a hyperbolioid or of a paraboloid, whichever it is, so that its internal surface is a minimum.

Solution by the PROPOSER.

(a). Let $PO=x$, $PG=y$, $\angle AOB=\angle DGE=2\pi/n$, $AO=r$, where O is the center of the circle circumscribing the larger base and G the center of the circle circumscribing the smaller base.



Then $DG=ry/x$, $AB=2r\sin(\pi/n)$, $DE=(2ry/x)\sin(\pi/n)$. Area $DGE=(r^2y^2/x^2)\sin(\pi/n)\cos(\pi/n)$.

$\therefore$ Area of upper base$=(nr^2y^2/x^2\sin(\pi/n)\cos(\pi/n)$.

$PC=\surd(PA^2-AC^2)=\surd[x^2+r^2-r^2\sin^2(\pi/n)$

$=\surd[x^2+r^2\cos^2(\pi/n)$.

Similarly $DF=(y/x)\surd[x^2+r^2\cos^2(\pi/n)]$.

Area $ADEB=r\sin(\pi/n)\surd[x^2+r^2\cos^2(\pi/n)][(x^2-y^2)/x^2]$.

The surface is the least when the upper or smaller base is the bottom of the cup.

Total surface of cup$=u$, volume$=c$.

$$\therefore\ u=nr\sin(\pi/n)\surd[x^2+r^2\cos^2(\pi/n)]\left(\frac{x^2-y^2}{x^2}\right)+\frac{nr^2y^2}{x^2}\sin(\pi/n)\cos(\pi/n)$$

$$c=\tfrac{1}{3}nr^2\sin(\pi/n)\cos(\pi/n)\left(\frac{x^3-y^3}{x^2}\right).\quad \text{Let } (r/x)\cos(\pi/n)=\tan\theta.$$

$$\therefore u=n\tan\theta\sec\theta\sec(\pi/n)\sin(\pi/n)(x^2-y^2)+ny^2\tan^2\theta\sec(\pi/n)\sin(\pi/n)\ .(1).$$

$$c=\tfrac{1}{3}n\tan^2\theta\sec(\pi/n)\sin(\pi/n)(x^3-y^3)\ \ldots.(2).$$

Differentiating (1) and (2) we get

$$\frac{dx}{dy}=\left(\frac{\sec\theta-\tan\theta}{\sec\theta}\right)\frac{y}{x}=(1-\sin\theta)\frac{y}{x}\ldots.(3). \qquad \frac{dx}{dy}=\frac{y^2}{x^2}\ldots.(4).$$

$$\frac{dx}{d\theta}=-\frac{(\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta}{2x\tan\theta}\ldots.(5).$$

$$\frac{dx}{d\theta}=-\frac{2\sec^2\theta(x^3-y^3)}{3x^2\tan\theta}\ldots.(6).$$

From (3) and (4) $y=x(1-\sin\theta)\ldots.(7)$.

From (5) and (6), $\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta=\dfrac{4\sec^2\theta(x^3-y^3)}{3x}$

This reduces to $(1+\sin^2\theta)(x^2-y^2)+2y^2\sin\theta=4(x^3-y^3)/3x\ldots.(8)$.

(7) in (8) gives $(3-8\sin\theta+3\sin^2\theta)\sin\theta=0$.
$\therefore\ \sin\theta=0$ or $\frac{1}{3}(4\pm\sqrt{7})$.
$\sin\theta$ cannot be zero nor greater than unity.
$\therefore\ \sin\theta=\frac{1}{3}(4-\sqrt{7})\ldots.(9)$.
$\therefore\ \tan\theta=\sqrt{\dfrac{4\sqrt{7}-7}{14}}\ldots(10)$.
From (7) and (9), $3y=x(\sqrt{7}-1)\ldots.(11)$.

(10) and (11) in (2) gives $x=\frac{1}{3}\left(\dfrac{2(38\sqrt{7}+89)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$.

From (11), $y=\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}\left(\dfrac{4(\sqrt{7}+13)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$.

$x-y=x-\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}x(4-\sqrt{7})=\left(\dfrac{2(\sqrt{7}-2)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$=altitude.

$r=x\tan\theta\sec(\pi/n)=\left(\dfrac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{6}}\sec(\pi/n)$.

$AB=2r\sin(\pi/n)=2\left(\dfrac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{6}}\tan(\pi/n)$, side lower base.

$DE=\dfrac{2ry}{x}\sin(\pi/n)=2\left(\dfrac{4(11\sqrt{7}-28)c^2}{49n^2\tan^2(\pi/n)}\right)^{\frac{1}{6}}\tan(\pi/n)$, side upper base.

(2). Let $y^2=4ax$ be the equation to the parabola, then we get from the Integral Calculus the two equations, between the limits x_2 and x_1,

$$u=\tfrac{8}{3}\pi\sqrt{a}[(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}]+4\pi ax_1\ldots.(1).$$
$$c=2\pi a(x_2^2-x_1^2)\ldots.(2).$$

From (1) and (2), by differentiation, we get

$$\frac{dx_2}{dx_1}=\frac{(x_1+a)^{\frac{1}{2}}-\sqrt{a}}{(x_2+a)^{\frac{1}{2}}}\ldots.(3)\,;\quad \frac{dx_2}{dx_1}=\frac{x_1}{x_2}\ldots.(4).$$

$$\frac{dx_2}{da}=-\frac{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}+3a(x_2+a)^{\frac{1}{2}}-3a(x_1+a)^{\frac{1}{2}}+3\sqrt{a}\,x_1}{3a(x_2+a)^{\frac{1}{2}}}\ldots.(5).$$

$$\frac{dx_2}{da}=-\frac{x_2^{\,2}-x_1^{\,2}}{2ax_2}\ldots.(6).$$

From (3) and (4) by eliminating dx_2/dx_1 we get

$$x_1=\frac{x_2^{\,2}-2x_2\sqrt{[a(x_2+a)]}}{x_2+a}\text{ and }x_1=0\ldots.(7).$$

Eliminating dx_2/da between (5) and (6) and substituting the first value of x_1 from (7) in the resulting equation, we get after reduction

$$36x_2^{\,3}-100ax_2^{\,2}-279a^2x_2-144a^3=0\ldots\ (8).$$

Let $a/x_2=z$ and (8) becomes

$$144z^3+279z^2+100z-36=0\ \ldots(9).$$

Let $z=v-\frac{31}{48}$ and (9) becomes

$$v^3-\tfrac{1283}{2304}v-\tfrac{8833}{55296}=0\ldots.(10).$$

$\therefore\ v_1=.861506,\ z_1=.215673.$
$v_2=-.445379,\ z_2=1.094212.$
$v_3=-.416138,\ z_3=-1.061971.$
$\therefore\ a=.215673x_2,\ -1.094212x_2$, or $-1.061971x_2$.

a cannot be negative. The first value of a gives $x_1=-.019813x_2$, a negative value and therefore not admissible. From this we learn that the second value of x_1 in (7), $x_1=0$ is the possible value and the cup is not a frustum of a paraboloid, but a paraboloid, a cup with a curved bottom.

$x_1=0$ in (5) and (6) $x_2^{\,2}-15ax_2+48a^2=0$.

$\therefore\ x_2=\dfrac{15\pm\sqrt{33}}{2}a.\quad \therefore\ x_2=\dfrac{15-\sqrt{33}}{2}a\ldots(11)$ is the admissible value.

From (11) and (2), $x_2=\left(\dfrac{(15-\sqrt{33})c}{4\pi}\right)^{\frac{1}{3}}$=altitude of cup.

$$a=\left(\frac{(43+5\sqrt{33})c}{3072\pi}\right)^{\frac{1}{3}}=\left(\frac{2c}{\pi(15-\sqrt{33})^2}\right)^{\frac{1}{3}}.$$

$y_2=2\sqrt{ax_2}=2\left(\dfrac{(15+\sqrt{33})c^2}{384\pi^2}\right)^{\frac{1}{6}}$, radius of top.

$$u=\tfrac{1}{96}\left(\frac{(1337+215\sqrt{33})4\pi c^2}{9}\right)^{\frac{1}{3}}[(3298-450\sqrt{33})^{\frac{1}{2}}-2].$$

MECHANICS.

116. Proposed by C. L. CHILTON, Greensboro, Ala.

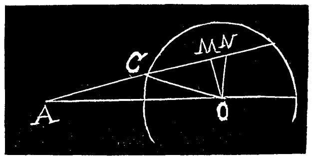
Given, the shaft ABC attached at one end by a pivot to the piston-rod of an engine (at A) and the other to a crank of the wheel CDE (at C). The shaft moves through the distance of two feet in one second from A to B and at the same time turns the crank from C to E. The force propelling the shaft along the constrained course from A to B is 5760 pounds. The mass of the rod and wheel and friction being not considered, what would be the kinetic energy of the machine? or the sum of the moment around O', the center of the wheel?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let AC, the connecting rod$=l$, $OC=r=$one foot, $\angle AOC=\theta$, force 5760 pounds along $AO=P$, component of P along $AC=Q$. Then moment of crank effect about $O=Q.OM$. In the right triangles AOM, AON, $AO:OM=AN:ON$.

$\therefore P:OM=Q:ON$.

$\therefore Q.OM=P.ON$.



Also $ON:OC=\sin OCN:\sin ONC$.

Let $\angle OAC=\varphi$. Then $\angle ONC=\frac{1}{2}\pi-\varphi$, $\angle OCN=\theta+\varphi$.

$$\therefore ON=\frac{r\sin(\theta+\varphi)}{\cos\varphi}=r\sin\theta+\frac{r\cos\theta\sin\varphi}{\cos\varphi}.$$

$$\text{but } \sin\varphi=\frac{r\sin\theta}{l}. \quad \therefore ON=r\sin\theta+\frac{r^2\sin\theta\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}.$$

$$\therefore \text{moment}=Pr\sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right).$$

Now θ varies from 0 to π.

$$\therefore \text{Average moment}=\frac{Pr\int_0^\pi \sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right)d\theta}{\int_0^\pi d\theta}=\frac{2Pr}{\pi},$$

a result independent of the connecting rod.

$2Pr/\pi=.6366Pr=3666.816$ or 3667 pounds.

117. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

How much lower must *one end* of a heavy uniform chain, wound round the circumference of a perfectly rough vertical wheel, hang than *the other end*, when the chain is on the point of motion?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let l=the entire length of the chain, a=the radius of the wheel, x=the longest part hanging down, $l-(\pi a+x)$=the shortest part, θ=the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right, T and T' the tensions at any points in those parts of the string to which x and $l-(\pi a+x)$ belong, and μ=the coefficient of friction.

We now have, from the usual theory, a unit of length of the chain being taken as a unit of weight.

$$T=Ce^{\mu\theta}+\frac{a}{1+\mu^2}[2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(1).$$

When $\theta=\frac{1}{2}\pi$, $T=x$, and (1) is

$$x=Ca^{\frac{1}{2}(\mu\pi)}+\frac{2\mu a}{1+\mu^2},\text{ or } C=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(2),$$

and (1) is $T=\left(x-\dfrac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta-\frac{1}{2}\pi)}+\dfrac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(3).$

In like manner, $T'=l-(\pi a+x)$ when $\theta=-\frac{1}{2}\pi$, and

$$T'=\left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta+\frac{1}{2}\pi)}+\frac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(4).$$

For equilibrium, $T=T'$ at the vertex, where $\theta=0$.

$$\therefore\left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\frac{1}{2}(\mu\pi)}=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(5),$$

giving $x=\dfrac{l-\pi a}{1+e^{-\mu\pi}}+\dfrac{2\mu a}{1+\mu^2}\ldots.(6).$

Then $2x+\pi a-l$ is found, the required length.

Also solved by *G. B. M. ZERR.*

AVERAGE AND PROBABILITY.

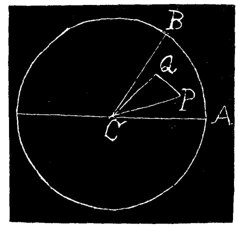
101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

By direct calculation obtain the average distance between two points in the surface of a circle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P, Q be the random points in the sector AOB. $AO=r$, $OP=y$, $OQ=x$,

$\angle AOB=\beta$, $\angle AOQ=\theta$, $\angle AOP=\varphi$. An element of the sector at P is $ydyd\varphi$; at Q, $xdxd\theta$. The limits of x, are 0 and r; of y, 0 and x; of θ, 0 and β; of φ, 0 and θ.



$PQ=u=[x^2+y^2-2xy\cos(\theta-\varphi)]^{\frac{1}{2}}$.

$$\therefore \triangle=\frac{\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxdxydyd\theta d\varphi}{\int_0^\beta\int_0^\theta\int_0^r\int_0^x xdxyd\theta d\varphi}$$

$$=\frac{16}{\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxyd\theta d\varphi dxdy=\frac{8}{3\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\{16\sin^3\tfrac{1}{2}(\theta-\varphi)$$

$$+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$

$$+3\sin^2(\theta-\varphi)\cos(\theta-\phi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}x^4d\theta d\varphi dx$$

$$=\frac{8r}{15\beta^2}\int_0^\beta\int_0^\theta\{16\sin^3\tfrac{1}{2}(\theta-\varphi)+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$

$$+4\sin^2(\theta-\varphi)\cos(\theta-\varphi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}d\theta d\varphi$$

$$=\frac{4r}{45\beta^2}\int_0^\beta\{48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+12\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-32\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-6\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$-64\cos\tfrac{1}{2}\theta+64+9\sin\theta\cos\theta+6\sin^3\theta\log(1+\operatorname{cosec}\theta)\}d\theta$$

$$=\frac{2r}{135\beta^2}\{16\sin^6\tfrac{1}{2}\beta+16\cos^6\tfrac{1}{2}\beta+96\sin^5\tfrac{1}{2}\beta+24\sin^4\tfrac{1}{2}\beta-12\cos^4\tfrac{1}{2}\beta-112\sin^3\tfrac{1}{2}\beta$$

$$-36\sin^2\tfrac{1}{2}\beta-720\sin\tfrac{1}{2}\beta+27\sin^2\beta+384\beta-4$$

$$-12(\sin^2\beta\cos\beta+2\cos\beta+2)\log(1+\sin\tfrac{1}{2}\beta)+12(\sin^2\beta\cos\beta$$

$$+2\cos\beta-2)\log\sin\tfrac{1}{2}\beta\}.$$

For the circle, $\beta=2\pi$, $\triangle=\dfrac{128r}{45\pi}$.

For the semicircle, $\beta=\pi$, $\triangle=\dfrac{256r}{45\pi}-\dfrac{1472r}{135\pi^2}$.

For the quadrant, $\beta=\frac{1}{2}\pi$.

$$\triangle=\frac{32r}{135\pi}\left(48+\frac{3}{\pi}-\frac{94\sqrt{2}}{\pi}-\frac{6}{\pi}\log\frac{1+\sqrt{2}}{2}\right).$$

II. Solution by the PROPOSER.

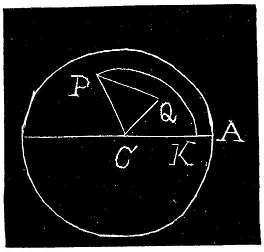
Take the center, C, of the circle and the horizontal radius CA as pole and initial line in polar coördinates.

Let $P(x, \theta)$ and $Q(y, \phi)$ be the random points. Then we have

$$PQ=\sqrt{x^2+y^2-2xy\cos(\theta-\phi)}.$$

Now with C as a center and a radius CP describe the arc PK.

It is evidently necessary to consider only those positions of the two points in which Q is confined to the sector PCK, since under this limitation PQ will take all possible distances.



An element of area of the circle at the point P is $xdxd\theta$, and at the point Q it is $ydyd\phi$.

The limits of y are 0 and x; of x, 0 and r; of ϕ, 0 and θ; and those of θ, 0 and 2π.

Consequently, we have for the required mean distance

$$M=\frac{1}{(\pi r^2)^2}\int_0^{2\pi}\int_0^{\theta}\int_0^{r}\int_0^{x}\sqrt{[x^2+y^2-2xy\cos(\theta-\phi)]}xdxd\theta.ydyd\phi$$

$$=\frac{4}{3\pi^2r^4}\int_0^{2\pi}\int_0^{\theta}\int_0^{x}\Big[8\sin^3\tfrac{1}{2}(\theta-\phi)+6\sin^3\tfrac{1}{2}(\theta-\phi)\cos(\theta-\phi)$$

$$-\tfrac{3}{2}\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)-1+\cos^2(\theta-\phi)\Big]x^4d\theta d\phi dx$$

$$=\frac{r}{15\pi^2}\int_0^{2\pi}\int_0^{\theta}\Big[8\sin\tfrac{1}{2}(\theta-\phi)+40\sin\tfrac{1}{2}(\theta-\phi)\cos^2\tfrac{1}{2}(\theta-\phi)$$

$$+6\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)$$

$$-48\sin\tfrac{1}{2}(\theta-\phi)\cos^4\tfrac{1}{2}(\theta-\phi)+3\cos2(\theta-\phi)-1\Big]d\theta d\phi$$

$$=\frac{2r}{45\pi^2}\int_0^{2\pi}\Big[32-32\cos\tfrac{1}{2}\theta-16\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+24\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+3\sin\theta\cos\theta$$

$$+3\sin^3\theta\log\left(\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)\Big]d\theta=\frac{128r}{45\pi}.$$

Professor Walker also furnished a second very excellent solution.

MISCELLANEOUS.

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91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.

The sides of this triangle can not all be correct because they are all *odd*.

Assuming that the *area* given is correct, it is required to determine the error in the sides.

I. Solution by M. A. GRUBER. A. M., War Department, Washington, D. C.

Knowing that every integral rational scalene triangle is the sum or the difference of two integral right triangles having equal altitudes, and knowing that 65 and 61 are hypotenuses of integral right triangles, we find that 65, 63, 16; 65, 60, 25; 65, 56, 33; and 65, 52, 39, are the respective sides of the right triangles having the hypothenuse 65; and that 61, 60, 11, are the sides of the only integral right triangle having the hypothenuse 61.

We observe that the triangles whose respective sides are 65, 60, 25, and 61, 60, 11, have the common leg 60. Taking 60 as the altitude, and finding the *sum* and the *difference* of the two triangles, we obtain two integral rational scalene triangles, sides being 65, 61, 36, and 65, 61, 14. The area of the former is $\sqrt{(81\times16\times20\times45)}=1080$, and of the latter, $\sqrt{(70\times5\times9\times56)}=420$.

$\therefore$ The side of the triangle in question should be 14.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Supposing the sides 61 and 65 to be right, but 21 to be wrong, we may put this side$=2a$, since it must be an even number. Thus, we get $(63+a)(63-a)(a+2)(a-2)=420^2$, whence $a^4-3973a^2=-192276$.

$\therefore a=\sqrt{\dfrac{3973\pm3975}{2}}$. Taking the lower sign, we obtain $2a=14$. Thus, the sides are 14, 61, 65. There may possibly be more rational values for the three sides to produce the area 420.

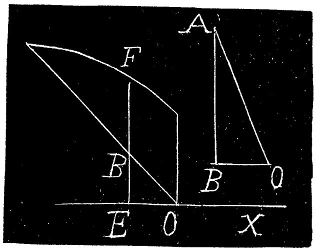
Also solved by *G. B. M. ZERR*.

92. Proposed by J. T. COLE, Columbus, O.

A staff $a=60$ feet high, casting a shadow on a horizantal plane due north $b=20$ feet long, falls due northeast. Find the area covered by the shadow.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O, B be the same points in both figures. OC the staff after it has fallen to the ground. OA the staff while falling.



Let $\angle AOB=\theta$, altitude of the sun$=\beta$, $OE=x$, $FE=y$. Then $\beta=\tan^{-1}(a/b)=\tan^{-1}(3)$, $AB=a\sin\theta$, $OB=a\cos\theta$, $OE=EB=(a/\sqrt{2})\cos\theta$, $BF=AB\cot\beta$.

$\therefore BF=a\sin\theta\cot\beta=b\sin\theta$.

$\therefore x=(a/\sqrt{2})\cos\theta$,

$y=x+b\sin\theta=x+(b/a)\sqrt{(a^2-2x^2)}=y'$.

Area passed over by the shadow$=CODFC$.

$\therefore A=\displaystyle\int_0^{a/\sqrt{2}}\int_x^{y'}dxdy=b/a\int_0^{a/\sqrt{2}}\sqrt{(a^2-2x^2)}dx$. $\therefore A=\frac{1}{8}\pi ab\sqrt{2}=150\pi\sqrt{2}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

139. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve neatly $\sqrt[4]{(m-x)}=\sqrt[4]{n}-\sqrt[4]{x}$.

140. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A man pays monthly $24.50 for 8 years for a loan of $1250. What is the rate %?

142. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wis.

How many teams of two horses each can a livery stable man send out who has 10 horses, assuming (1) that we consider the way the team is hitched and (2) that we do not.

Suppose he has 8 horses; 10 horses. Suppose he has 7 buggies, then how many different rigs can he send out, assuming that he has 10 horses, and counting both one and two horse rigs?

[We will publish a series of problems of this character in future numbers of the MONTHLY.]

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

GEOMETRY.

169. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Theorem. Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [From *Olney's Geometry*, Section VIII, Proposition XIV.]

1. Required the proof. 2. Is this proposition found in any other text-book of Geometry ?

170. Proposed by CHARLES C. CROSS, Whaleyville, Va.

If p, q, r are the distances of the orthocenter from the sides, prove that

$$4\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=\left(\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right)\left(\frac{b}{q}+\frac{c}{r}-\frac{a}{p}\right)\left(\frac{c}{r}+\frac{a}{p}-\frac{b}{q}\right).$$

171. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the nearest distance of the parabola $y^2=16x$ and the ellipse $16x^2+9y^2-160x-144y+832=0$.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Integrate $\int\frac{\sqrt{(1+y)}}{1+y^2}dy$.

134. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the curve for which the sum of that part of the tangent, lying between the point of contact and the axis of abscissas, and the corresponding ordinate is constant $=c$, and which passes through the point (a, b).

⁂ Solutions of these problems should be sent to J. M. Colaw not later than August 10.

MECHANICS.

124. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

At one end of a weightless thread of length l is fastened a sphere of radius r, and the other end of the thread is fastened to a vertical axis. The axis is put into motion of constant angular velocity ϕ. What is the maximum angle which the thread will make with the vertical axis?

⁂ Solutions of this problem should be sent to B. F. Finkel not later than August 10.

DIOPHANTINE ANALYSIS.

87. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find three numbers in arithmetical progression the sum of whose cubes is a cube.

88. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find three square numbers in harmonical progression.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than August 10.

AVERAGE AND PROBABILITY.

110. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of the triangle formed by joining three random points taken on the surface of a regular hexagon, two on one side of a diagonal and the third on the other side.

111. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

If a radius be drawn at random in a given semi-circle, and a point taken at random in one of the sectors formed, show that the chance that a random line drawn through the point will cut the arc of the sector is

$$1-\frac{1}{\pi^2}\log 2.$$

112. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Two circles are drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle. Show that the chance of their both resting on the same diameter of the given circle is

$$\frac{4}{\pi}(8\log 2-5).$$

⁂ Solutions of these problems should be sent to B. F. Finkel not later than August 10.

MISCELLANEOUS.

110. Proposed by E. W. MORRELL, South Tunbridge, Vt.

If a and b be the sides of a triangle, A and B the angles opposite, then will $\log b-\log a=\cos 2A-\cos 2B+\frac{1}{2}(\cos 4A-\cos 4B)+\frac{1}{3}(\cos 6A-\cos 6B)+\ldots.$

111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Exhibit $\cos^3\theta\sin^3\theta\sin^2\phi\cos\phi$ as a series of harmonics.

112. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

(a) Find the area enclosed by four circles, of which two touch the x-axis, and two the y-axis, at the origin.

(b) Required the area enclosed by four parabolas, of which two touch the x-axis, and two the y-axis, at the origin.

113. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Deduce the Sylvestrian Reciprocant from $x^4+y^4=4x^2y^2$.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

NOTES.

Dr. W. B. Fite has been appointed Instructor in Mathematics at Cornell University. He will take most of the advanced work formerly done by Dr. G. A. Miller.

Professor Alexander S. Chessin has been elected Professor of Mathematics in the Washington Uuiversity, St. Louis. The University is to be congratulated in securing such an able man.

Mr. Paul A. Towne, A. M., has sent us a number of very interesting magic squares constructed on the sides and hypothenuse of right triangles, but owing to the difficulty in reproducing them, we must omit their publication.

Brief notice will be made in the next issue of the MONTHLY of the following books: *One Hundred Problems in Mathematical Physics*, by E. P. Thompson; *Atoms and Energy*, by D. A. Murray; *Experimental Physics*, by Eugene Lommel; *An Introduction to the Study of Chemistry*, by Ira Remsen.

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BIOGRAPHY.

ISAAC NEWTON.

By VIRGINIA J. CRAIG, A. B., Springfield, Mo.

Sir Isaac Newton was born at Woolthorpe in Lincolnshire in 1642, the year of Galileo's death. He first attended the village school and later the public school at Grantham. He was a delicate child and at first far from industrious. An unprovoked attack from a boy above him led to a fight in which Newton's pluck was victorious. This success led him to greater exertions in school, and after a time he rose to be head boy. He early displayed a taste for mechanical inventions. He made wind-mills, water-clocks, kites, dials, and a carriage propelled by the rider. When he had attained his fifteenth year, his mother took him home to assist her in the management of the farm, but his dislike for farming and desire for study induced her to send him back to Grantham, where he remained until his eighteenth year, when he entered Trinity college, Cambridge. Little is known as to his attainments at this time. He tells us that he had bought a book on astrology at a fair, but on account of his ignorance of trigonometry could not understand its figures. So he bought a Euclid, but on looking it over, thought the propositions self evident and laid it aside as a trifling work.

He now applied himself to Descartes' Geometry, which he mastered without assistance. This work inspired him with a love for the higher mathematics. He tells us that in 1665 he found the method of Infinite Series and computed

the area of the Hyperbola to 52 figures. He did not communicate his invention to his friends till 1669, when he placed in the hands of Barrow a tract *De Analysi per Æquationes Numero Terminorum Infinitas*. Supposing the abscissa to increase uniformly in proportion to the time he looked upon the area of a curve as a nascent quantity increasing by continued fluxion in the proportion of the length of the ordinate. The expression obtained for the fluxion he expanded into a finite or infinite series of monomial terms. Had this tract been published then instead of 42 years later there would probably have been no occasion for that long and deplorable controversy between Newton and Leibnitz.

Newton received the degree of B. A. in 1665 and two years later was made a fellow of Trinity College. He gave much study to lenses and prisms at this time. In 1669 he was elected Lucasian professor and soon after admitted to the Royal Society. At the meeting at which he was elected to the Society a description of a reflecting telescope which he had invented was read. Newton published an account of his discovery of the spectrum. Yet he made the mistake of supposing that all prisms would give a spectrum of the same length. His publication involved him in a controversy painful to him. Although wearied and resolved to publish nothing more, he did not, fortunately, stick to this resolution. He explained the color of thin and thick plates and the inflexion of light and wrote on double refraction, polarization and binocular vision. He invented a reflecting sextant for observing the distance between the moon and fixed stars.

It is supposed that it was at Woolthorpe in 1666 that Newton's thoughts were directed to the subject of gravity. Tradition marked a tree till 1820, as that from which the apple fell when the tree was cut down and its wood preserved.

Kepler had proved that each planet revolves in an elliptical orbit round the sun, whose centre occupies one of the foci of the orbit, that the radius vector of each planet describes equal areas in equal times and that the squares of the periodic times of the planets are in the same proportion as the cubes of their mean distances from the sun. Newton came to believe from analogy of bodies falling to the earth that gravitation was the cause of the moon's remaining in its orbit. By calculating from Kepler's laws Newton had already proved that the force of the sun acting upon the different planets must vary as the inverse square of the distances of the planets from the sun. He now found that the moon was deflected from the tangent in every minute through thirteen feet. But observing the distance through which a body would fall in one second on the earth, and calculating on force diminishing as the inverse square, he found that the earth's attraction would deflect the moon fifteen feet. Newton was dissatisfied with the discrepancy. A little later, however, when Picard made a more accurate estimate of the earth's magnitude, Newton eliminated the discrepancy.

While as yet the law of inverse squares was not regarded established, Newton proved that according to this law a body would travel round the sun in an ellipse.

In 1685 and 1686 Newton composed almost the whole of his great work, the *Principia*. The credit of prior recognition of the law of the inverse squares was claimed by Hooke and Newton was generous and just enough to allow the claim. The whole work was published in 1687.

A little later Newton's health became quite poor. He had neglected so often to take food and sleep and for quite a time suffered evil consequences.

He had taken an active part in protecting the university against the encroachments of the crown, and this fact was the cause of his election to parliament as a representative of the university. During his London residence he became a friend of Locke.

Newton was now in his fifty-fifth year, and up to this time had received no mark of national gratitude. Through Montague's efforts, he was now given the wardenship and later the mastership of the mint.

Up to 1687 Newton's method of fluxions was still a secret. One of the most important rules of the method forms the second lemma of the second book of the *Principia*. Yet Newton did not exhibit his method in the results. So it was not communicated to the scientific world until 1693 in the second volume of Dr. Wallis' works.

Newton's admirers in Holland had informed Dr. Wallis that Newton's method of fluxions passed there under the name of Leibnitz's *Calculus Differentialis*. It was therefore thought necessary that an early opportunity should be taken of asserting Newton's claim to be the inventor of the method of fluxions, and this was the reason for the method first appearing in Wallis' work. A further account of the method was given in Newton's *Optics*.

There is no doubt as to Newton being the inventor of fluxions, but it has been strongly contested whether Leibnitz invented his calculus independently, or borrowed it from the fluxional calculus with which at bottom it is identical. In 1674 Leibnitz announced to the Royal Society that he possessed analytical methods depending on infinite series by which he had found theorems of great importance relating to the quadrature of the circle. In reply he was informed that Newton had discovered similar methods for the quadrature of curves which extended to the circle.

In 1676 Newton had sent to Leibnitz a letter containing his binomial theorem, the now well known expressions for the expansion of an arc in terms of its sine, and its converse that of the sine in terms of the arc. This letter also contained an expression in an infinite series for the arc of an ellipse. Inquiries from Leibnitz followed and a reply by Newton. Newton commenced his letter by commending the method of Leibnitz for the treatment of series. He then states his three methods but does not clearly explain them. Leibnitz in reply explained his method of drawing tangents to curves, introducing his notation dx and dy for the infinitely small differences of the successive coordinates of a point on the curve, and showed that his method could be readily applied if the equation contained irrational functions. Further on he gave one or two examples of problems involving the integration of a differential equation of the

first order which shows that Leibnitz was then in possession of the principles of the integral calculus. The sign of integration has been found to have been employed by him in a manuscript of October 29, 1675, preserved in the royal library of Hanover. This proves that Leibnitz was in possession of his method before he had received any account of Newton's method of fluxions. In 1684 Leibnitz made his method public. Thus while Newton's claim to priority of discovery is admitted by all, Leibnitz was the first to publish his method. Insinuations were made in 1699 that Leibnitz had derived his whole method from Newton and had merely changed the name and notation. At first Newton recognized Leibnitz as an independent discoverer of the calculus, and tried to stop the attack on Leibnitz. Yet he felt the justice of the recognition of his own priority, and against his will was dragged into a discussion which continued long after his death. The bitterness of the discussion was greatly augmented by national emulation. All mathematicians are now agreed that both men are entitled to be regarded independent discoverers of the principles of the calculus, while Newton was master of the method of fluxions before Leibnitz discovered his method.

In 1707 Whiston published the algebraical lectures which Newton had delivered at Cambridge.

In addition to these other mathematical works, Newton had solved two celebrated problems proposed by Bernoulli and Leibnitz. In June, 1696, Bernoulli addressed a letter to the mathematicians of Europe challenging them to solve two problems: (1) To determine the brachistrochrone between two given points not in the same vertical line. (2) To determine a curve such that if a straight line drawn through a fixed point A meet it in two points P_1 and P_2, then $AP_1^m + AP_2^m$ will be constant. Six months were allowed by Bernoulli for the solution of the problems, and in the event of none being sent to him he promised to publish his own. The six months elapsed without any solution being produced; but he received a letter from Leibnitz stating that he had "cut the knot of the most beautiful of these problems," and requested that the period of their solution should be extended to Christmas next. This was done. On January 29th, 1696-97, Newton received two copies of the problems, and on the following day gave a solution of them to Montague, then president of the Royal Society. He announced that the curve required in the first problem must be a cycloid, and he gave a method of determining it. He solved also the second problem and showed that by the same method other curves might be found which shall cut off three or more segments having the like properties. Solutions were also obtained from Leibnitz and the Marquis de L'Hospital, yet Bernoulli recognized the author in his disguise; "tamquam," says he, "ex ungue leonem."

In 1699 Newton's position as a mathematician and natural philosopher was recognized by the French Academy of Sciences. Eight foreign associates were added, among whom were Leibnitz and Newton.

In 1703 Newton was elected president of the Royal Society, and annually

re-elected during the remainder of his life. He thus held the office for twenty-five years. Prince George of Denmark, the queen's husband, who was a fellow of the Royal Society, was deeply impressed by Newton's genius. The queen, accordingly, wished to honor her greatest subject. In April, 1705, she held court at Trinity lodge and conferred knighthood upon Newton. At the court of George I. Newton was a very popular visitor.

From an early period of his life Newton had paid great attention to theological studies, and it is well known that he had begun to study the prophecies before 1690. M. Biot, with a view of showing that his theological writings were the production of his dotage, fixed their date between 1712 and 1719. That Newton's mind was even then quite clear and powerful is sufficiently proved by his ability to attack the most difficult mathematical problems with success. For in 1716 Leibnitz proposed a problem for solution "for the purpose of feeling the pulse of English analysts." The problem was to find the orthogonal trajectories of a series of curves represented by a single equation. Newton received this problem about 5 o'clock in the afternoon, but, though fatigued with business, he solved the problem the same evening.

He left a number of biblical and theological dissertations.

After a painful illness endured with great patience, he died in the eighty-fifth year of his age, on March 20th, 1726.

In preparing this sketch the following works have been consulted: Cantor's *Gaschicte der Mathematik*: Ball's *A Short History of Mathematics*; Cajori's *History of Mathematics*, and the *Encyclopedia Britannica*.

ON THE UTILITY OF STUDYING NON-EUCLIDEAN GEOMETRY.

By P. BARBARIN.

I. The question of the foundation of the theory of parallels has been one of the most interesting scientific preoccupations of this century; it has caused to gush forth torrents of works and given subject to remarkable researches.

Thanks to the theorems of Legendre, to the works of the two Bolyai, of Lobachevski, and of Riemann, of Poincaré, Flye St. Marie, Klein, De Tilly, etc., we cannot any more be deceived on the true import of the celebrated proposition which bears the name of *Postulate of Euclid.*

1st. This is not in any way contained in the classical definitions of the straight and the plane;

2d. This is, among three hypotheses equally admissible, and which cannot all be rejected, only the most simple.

Is it perhaps the single case which has given to the grand Greek geometer

the choice of his system of geometry ? or has he perceived, at least in part, the difficulties and the greater theoretic complication of the other two ? We shall never know with certainty. But in the presence of his work so perfect and so rigorous, one thing appears however to be beyond doubt: the post which he assigned to his postulate, the enunciation which he gave of it, attest that this proposition had in his eyes only the value of an hypothesis ; without that, he would have formulated it in other terms and would have attempted to demonstrate it.

The ideas of Lobachevski and of Riemann were diffused only very slowly. They were spread above all thanks to the translations of Houël. This scientist, whose activity and power of work were prodigious, could not resist the desire to learn all the European languages with the aim of being able to read in their original text, and then to make known to his contemporaries, the most celebrated mathematical works. He admired Lobachevski, whom he surnamed *the modern Euclid*, and in his course professed at the scientific faculty of Bordeaux, he did not neglect any occasion to put him in evidence.

II. Houël was persuaded that the knowledge of the non-Euclidean geometry is indispensable for possessing to the bottom the mechanism of the Euclidean geometry. Notwithstanding its paradoxical form, this idea is most just.

General geometry or *Metageometry* contains in fact a great number of propositions common to all the systems, and which should be enunciated with the same terms in each of these.

If the general proposition can be demonstrated also in these general terms, these ought to be preferred even when, to reach this, it is necessary to subject the ordinary form to some modifications.

To cite only one example, we take the convex quadrilateral inscribed in a circle.

In Euclidean geometry, *the sum of two opposite angles is constant and equal to two right angles ;* in non-Euclidean geometry *this sum is variable.* Notwithstanding that, the two forms may be conciliated, since in the two cases *the sum of two opposite angles equals that of the other two*, and this is sufficiant to make a convex qualrilateral inscriptible.

Confronting the proposition with that concerning the circumscribed quadrilateral, we put in full light a correlation which *a priori*, ought evidently to exist.

This correlation, which is the foundation itself of general geometry, and which does not always appear in the ordinary geometry with the same clearness, can be utilized for finding new properties of the figures.

EXAMPLE. *Every conic is the locus of points such that the sum of the tangents from these drawn to two circles is constant ;* every conic will then be also *the curve envelope of the straights which cut two given circles under angles of which the sum is constant.* (Excellent problem for direct investigation).

III. Is it proper to associate non-Euclidean geometry with teaching, and in what measure ?

If it is a question of higher instruction, with ardor we answer affirmative-

ly. In the courses of higher geometry in the Universities the names of Bolyai, Lobachevski, Riemann have their assigned place, and there are yet divers unexplored domains on the road which these scientists have opened.

In so far as referring to the secondary instruction, the question is more delicate. The programs of the preparatory courses of the higher schools contain all, or almost all. special mathematics and spherical geometry. It would not be then a great inconvenience to make there from time to time a discrete allusion to general geometry: on the contrary, the attention of the pupils and their critical spirit would be kept awake by the necessity of investigating if the special proposition which is expounded to them be of order particular or general.

Two indispensable conditions alone should be satisfied; it is requisite:

1st. *That in all the books put in the hand of the pupils, the hypothetical character of the postulate of Euclid should be well put in relief.*

In my classes I recur with success to the simple proceeding which follows, and which I recommend. Take the straight AB and the two equal perpendiculars AC, BD: the angles ACD, BDC are equal, and may be right, acute, or obtuse.

But whichever may be that among these three hypotheses which we assume for this particular quadrilateral, we must conserve it for *all* the other like quadrilaterals. We choose the system of geometry in which these are right, and which corresponds to the Euclidean hypothesis.

2d. *That the invertibility of the postulate of Euclid be cut out of all the demonstrations in which it can be done without, and where however it is wrongly used.* See, for example, the theorem on the sum of the faces of a trihedral or polyhedral angle.

We should recognize that many efforts have been made in these latter years, in the sense indicated. If the notions of general geometry tend to become popular, the honor of it is due above all to the periodicals which have given their hospitality, and in special mode to *Mathesis*, so well directed by our excellent confrere *P. Mansion* of Gand.

In the course of the last eight or ten years this journal has published numerous articles on metageometry written with equal competence and good sense. We advise students to read them.

[Written by P. Barbarin for *Le Matematiche*, and translated by the English Editor G. B. Halsted].

REDUCED NUMBERS.

By A. LATHAM BAKER, Ph. D., University of Rochester, Rochester, N. Y.

The facts of this article are not new, but the presentation is novel and much more direct and simple than any other with which I am acquainted. The

steps are *suggestive* and therefore valuable from a pedagogical point of view, enough so, I hope, to warrant their publication.

The complex number $z=x+iy$ can be considered as the root of a quadratic equation

$$Az^2 \pm Bz + C=0,$$

in which A and C must have the same sign, since

$$z=\frac{\pm B}{2A} \pm \sqrt{\frac{B^2-4AC}{4A^2}}$$

requires for complex roots $B^2-4AC=-n^2$.

$\because AC=\frac{B^2+n^2}{4}>0$, and A and C must have the same sign, n being any real number.

The position of z in the Argand plane will evidently depend upon the relative values of A, B, C.

Denoting the values of A, B, C as in the following table, where a, b, c are integers, we have, according to the relative size of A, B, C, the following six cases with the corresponding roots :

	A	B	C	$z=x+iy$
1	a	$a+b$	$a+b+c$	$\frac{a+b}{2a}+i\sqrt{\frac{a+b+c}{a}-\frac{(a+b)^2}{4a^2}}$
2	a	$a+b+c$	$a+b$	$\frac{a+b+c}{2a}+i\sqrt{\frac{a+b}{a}-\frac{(a+b+c)^2}{4a^2}}$
3	a	$a-b$	$a+c$	$\frac{a-b}{2a}+i\sqrt{\frac{a+c}{a}-\frac{(a-b)^2}{4a^2}}$
4	a	$a-c-b$	$a-b$	$\frac{a-b-c}{2a}+i\sqrt{\frac{a-b}{a}-\frac{(a-c-b)^2}{4a^2}}$
5	a	$a-b$	$a-b-c$	$\frac{a-b}{2a}+i\sqrt{\frac{a-b-c}{a}-\frac{(a-b)^2}{4a^2}}$
6	a	$a+b$	$a-c$	$\frac{a+b}{2a}+i\sqrt{\frac{a-c}{a}-\frac{(a+b)^2}{4a^2}}$

In forms 1 and 2, $|x| \geqq \frac{1}{2}$, $x^2+y^2 \geqq 1$,

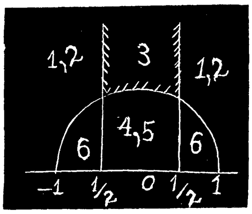
In form 3, $|x| \leqq \frac{1}{2}$, $x^2+y^2 \geqq 1$.

In forms 4 and 5, $|x| \leqq \frac{1}{2}$, $x^2+y^2 \leqq 1$.

In form 6, $|x| \geqq \frac{1}{2}$, $x^2+y^2 \leqq 1$.

Hence the roots of the different forms are located in the regions corresponding to the numbers shown in the diagram.

Points in 1,2 can be transformed into points in 1,2 by a unitary substitution



$$w=\frac{az+b}{cz+d}\quad [ad-bc=1]\quad \text{viz., } w=z+b.$$

Points in 6 can be transformed into points in 6 by the unitary substitution $w=\dfrac{z+b}{z+b+1}$, and points in 4, 5 can be transformed into points in 4,5 by the substitution $w=\dfrac{-1}{z+d}$. That is, *in any one of the five regions of the plane except* 3, *any point can be transformed into some other point of the same region* by a unitary substitution. This will be found to be impossible in 3, thus marking off 3 as a unique region, accordingly called the *fundamental triangle.*

Numbers (or points) connected by a unitary substitution are called *equivalent numbers or points.*

We shall find that all points of the plane can be reduced to some point in the fundamental triangle, and hence the points in the fundamental triangle are called *reduced points.* Also that no two reduced points can be equivalent.

1°. *Every point at a finite distance above* the x axis is equivalent to one and only one reduced point.*

Translation simply, $z+b$, will carry the point to within the strip bounded by $-\frac{1}{2}$, $+\frac{1}{2}$, either into the fundamental triangle or else below it, so that we need only consider points within the region 4,5. For points in this region put $w=\dfrac{-1}{z}=\mu+i\nu$, where $\nu=\dfrac{y}{x^2+y^2}>y$, unless $x^2+y^2=1$.

If $x^2+y^2=1$, z or w is reduced. one or the other.

If otherwise, translate w horizontally to within the strip and repeat the previous operation of inversion, and so on until the translation carries the point within the fundamental triangle. But since $\nu>y$, each inversion raises the point in the plane so that eventually it will become high enough to be carried by horizontal translation into the fundamental triangle.

That it will not require more than a finite number of these inversions can be seen as follows :

The inversion of a point z in the region 4, 5 raises it from the point whose

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*[Since $w=\dfrac{az+b}{cz+d}=\dfrac{(az+b)(cx+d-iy)}{(cx+d)^2+c^2y^2}+i\,\dfrac{y}{(cx+d)^2+c^2y^2}=\mu+i\nu$, ν and y have the same sign and only the upper portion of the Argand plane need be considered; that is, *equivalent points are on the same side of the x axis.*

ordinate is y to one whose ordinate is $\nu=\frac{y}{x^2+y^2}=\frac{y}{r^2}$, A translation and in version changes the distance to $\frac{y}{r^2 r_1^2}$, and so on until the final distance above the x axis is $\frac{y}{r^2 r_1^2 \dots r_n^2}$, after $n+1$ operations.

If $\frac{y}{r^2 r_1^2 \dots r_n^2}>1$, the point has reached a distance where horizontal translation will carry it within the fundamental triangle. If we call the largest of the r's, R, then under the most unfavorable circumstances, $(m>n)$

$$\frac{y}{R^{2m}}>1 \text{ or } y>R^{2m}, \text{ or } m\log R^2<\log y, \text{ or } m<\frac{\log y}{\log R^2}.$$

But both the elements of the fraction are finite, and m must be finite.

But $n<m$, and is therefore finite. Q. E. D.

2°. *No two reduced points can be equivalent.*

Suppose $w=\frac{az+b}{cz+d}$ to be the equivalent points, z being reduced.

Then $w=\frac{(az+b)(cx+d-iy)}{(cx+d)^2+c^2y^2}+i\frac{y}{(cx+d)^2+c^2y^2}$, and $\nu=\frac{y}{c^2(x^2+y^2)+2cdx+d^2}$

But if z is reduced, $x^2+y^2>1$ and $|2x|<1$; hence

$$\begin{aligned} c^2(x^2+y^2)+2cdx+d^2 &> c^2+d^2+2cdx \\ &> c^2+d^2-cd \\ &> 1 \end{aligned}$$

unless $c=0$, $d=\pm 1$, or $c=\pm 1$, $d=0$.

If $c=0$, $d=\pm 1$, then $w=z+b$, and w is outside the fundamental triangle.

If $c=\pm 1$, $d=0$, then $w=-1/z$, $(\mu^2+\nu^2)(x^2+y^2)=1$, and since $x^2+y^2>1$, $\mu^2+\nu^2<1$, and w is not reduced.

In the other cases, $\nu<y$, and *a reduced point is lowered in the plane by a unitary substitution of the form* $w=-1/z$.

If now w is a reduced point, then the unitary transformation $z=\frac{dw-b}{-cw+a}$ should give $y<\nu$, but this cannot be since this is the inverse substitution of $w=\frac{az+b}{cz+d}$ and we have already found $\nu<y$. Hence w is not a reduced point. Q. E. D.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

144. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

A hired a house for one year for \$300; at the end of four months he takes in M as a partner; and at the end of eight months he takes in P. At the end of the year what rent must each pay? [From Greenleaf's *National Arithmetic*, page 442.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; and J. W. DAPPERT, C. E., Taylorville, Ill.

1st Method. \$300 for one year is \$25 per month. A has the house alone for four months at \$25=\$100. He shares it with M for four months at \$12½ each =\$50. With M and P for four months at \$8⅓ each=\$33⅓.

$\therefore$ A pays 100+50+33⅓=\$183⅓.

M pays 50+33⅓=\$83⅓.

P pays \$33⅓. DAPPERT.

2nd Method. A, 12 months+M, 8 months+P, 4 months=24 months for one person.

\$300÷24=\$12½ per month.

12×12½=\$150, what A pays.

8×12½=\$100, what M pays.

4×12½=\$50, what P pays.

By the second method the house rents for \$12½ per month for four months, \$25 per month for four months, and \$37½ per month for four months.

The modern idea would call the first method the correct one.

Also solved by *W. P. WEBBER*, and *H. C. WHITAKER*.

NOTE.—Problems of this nature have been the subject of much discussion in the past. The first method is unquestionably the correct one, as a person is required to pay rent for the time he occupies the house. Coach problems, where A hires a coach for a certain sum and then on the way takes in B, and then a little farther on takes in C, what amount should each pay? should be solved on the basis that each pays in proportion to the distance he rides, such problems being of the same nature as the one under discussion. ED. F.]

145. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield. Mo.

By discounting a note at 20% per annum, I get 22½% per annum interest; how long does the note run? [From Ray's *Higher Arithmetic*, page 405.]

Solution by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md., and W. P. WEBBER, Houston, Miss.

Face×Rate×Time (in years)=Discount....(1).

Proceeds×rate×Time (in years)=Interest....(2).

In this case, the discount equals the interest; hence, $P\times r\times T=F\times R\times T$.

Cancelling T and substituting the two rates, $P\times\frac{9}{40}=F\times\frac{1}{5}$.

$\therefore\ P=\frac{8}{9}F.$

Assuming \$100 for the face of the note, the proceeds will be \$$88\frac{8}{9}$, and the discount or interest \$$11\frac{1}{9}$.

From (2), Time$=\dfrac{\text{Int.}}{P\times r}=\dfrac{11\frac{1}{9}}{88\frac{8}{9}\times\frac{9}{40}}=\frac{5}{9}$ year$=200$ days.

Also solved by *G. B. M. ZERR*, and *H. C. WHITAKER*.

ALGEBRA.

119. Proposed by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

Given $\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{3\times5}+\dfrac{17x^7}{3^2\times5\times7}+\dfrac{62x^9}{3^2\times5\times7\times9}\ldots\ldots$

Find the general term and interval of convergence of this series.

I. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Using the notation of the Calculus, let $\tan x$ be represented by $f(x)$ or simply f. Then dy/dx will be $f'(x)$ or more simply f'; similarly, f'', f''', f^{iv}, f^{v}, etc., will be used. We have then by differentiating :

$$f(x)=\tan x \qquad\qquad f(0)=0.$$

$$f'=\sec^2 x \qquad\qquad f'(0)=1.$$

$$f''=2\sec^2 x\tan x=2ff' \qquad\qquad f''(0)=0.$$

$$f'''=2ff''+2f'^2 \qquad\qquad f'''(0)=2.$$

$$f^{\text{iv}}=6f'f''+2ff''' \qquad\qquad f^{\text{iv}}(0)=0.$$

$$f^{\text{v}}=6f''^2+8f'f'''+2ff^{\text{iv}} \qquad\qquad f^{\text{v}}(0)=16.$$

$$f^{\text{vi}}=20f''f'''+10f'f^{\text{iv}}+2ff^{\text{v}} \qquad\qquad f^{\text{vi}}(0)=0.$$

$$f^{\text{vii}}=20f''+30f'f^{\text{iv}}+12f'f^{\text{v}}+2ff^{\text{vi}} \qquad\qquad f^{\text{vii}}(0)=272.$$

We see that the *even* differential coefficients vanish and the *odd* follow a remarkable law : The first term is the middle term in the expansion by the Binomial Theorem and the following terms are the double of the successive coefficients of the Binomial Theorem.

By this law we can write any of the odd derivatives, thus :

$$f^{\text{ix}}=70f^{\text{iv}2}+112f'''f^{\text{v}}+56f''f^{\text{vi}}+16f'f^{\text{vii}}+2ff^{\text{viii}} \qquad\qquad f^{\text{ix}}(0)=7936.$$

$$f^{\text{xi}}=252f^{\text{v}2}+420f^{\text{iv}}f^{\text{vi}}+240f'''f^{\text{vii}}+90f''f^{\text{viii}}+20f'f+2^{\text{ix}}ff^{\text{x}} \qquad\qquad f^{\text{xi}}(0)=353792.$$

$$f^{\text{xiii}}=924f^{\text{vi}2}+1584f^{\text{v}}f^{\text{vii}}+990f^{\text{iv}}f^{\text{viii}}+440f'''f^{\text{ix}}+132f''f^{\text{x}}+24f'f^{\text{xi}}+2ff^{\text{xii}}$$

whence $f^{\text{xiii}}(0)=22368256$, and thus indefinitely.

That is, in writing f^{xiii}, I write the coefficients of $(a+b)^{12}$, which are 1, 12, 66, 220, 495, 792, 924, etc. I set down 924 and double each of the others and get the result as given above.

Substituting the values obtained in MacLaurin's Formula, we have :

$$\tan x=x+\frac{2x^3}{3\,!}+\frac{16x^5}{5\,!}+\frac{272x^7}{7\,!}+\frac{7936x^9}{9\,!}+\frac{353792x^{11}}{11\,!}+\frac{22368256x^{13}}{13\,!}+\text{etc.}$$

$$=x+\frac{x^3}{3}+\frac{2x^5}{3\times5}+\frac{17x^7}{3^2\times5\times7}+\frac{62x^9}{3^2\times5\times7\times9}+\frac{1382x^{11}}{3^2\times5^2\times7\times9\times11}$$

$$+\frac{21844x^{13}}{3^3\times5^2\times7\times9\times11\times13}.$$

II. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

The general term of the series is

$$\frac{2^{2n}(2^{2n}-1)}{2n\,!}Bx^{2n-1},$$

where B (Bernoulli) stands for the coefficient of x^n in the expansion of

$$\frac{x}{e^x-1}=\frac{x}{\dfrac{x}{1}+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dfrac{x^4}{4}+\text{etc.}}$$

120. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A hollow sphere has within it a solid sphere; a quantity of water equal to $1/m$ of the capacity of the hollow sphere is poured in and just covers the solid sphere. Prove that there are two solid spheres, either of which answers the conditions; also find the maximum value $1/m$, beyond which the question is not possible.

I. Solution by J. A. COLSON, Searsport, Me.

The volume of the inner sphere (radius r)+the volume of the water=the volume of a segment whose radius is R and whose height is $2r$; that is

$$\tfrac{4}{3}\pi r^3+\frac{4}{3m}\pi R^3=\tfrac{1}{3}\pi(2r)^2(3R-2r),$$

which reduces to $r^3-Rr^2+R^3/3m=0\ldots.(1)$.

Hence, by Cardan's Method one value of r is

$$r_1=\tfrac{1}{3}R+\sqrt[3]{\left[\tfrac{1}{2}\left[-\frac{R^3}{27}\left(\frac{9}{m}-2\right)+\frac{R^3}{9}\sqrt{\left(\frac{9}{m^2}-\frac{4}{m}\right)}\right]\right]}$$

$$+\sqrt[3]{\left[\tfrac{1}{2}\left[-\frac{R^3}{27}\left(\frac{9}{m}-2\right)-\frac{R^3}{9}\sqrt{\left(\frac{9}{m^2}-\frac{4}{m}\right)}\right]\right]}\ldots.(2).$$

Now (1) will have either one or three real roots. If three, two of them by Descarte's Rule of Signs will be positive and one negative. In the former case the quantity $9/m^2-4/m$ under the radical sign in (2) will be positive. In the latter case it will be negative. Consequently the point of change from the reducible to the irreducible case of Cardan will be when $9/m^2-4/m=0$, which gives $1/m=\frac{4}{9}$. Substituting this value in (1) we have $r^3-Rr^2+4R^3/27=0$..(3), whose roots are $r_1=-R/3$, and $r_2=r_3=\frac{2}{3}R$. The positive roots being equal shows that $\frac{4}{9}$ is the maximum value of $1/m$ which will give positive roots.

II. Solution by the PROPOSER.

Let a be the radius of the hollow sphere, and x the diameter of the solid sphere, and, therefore, the weight of the water. Then the center of the hollow sphere is $x-a$ or $a-x$ from the surface of the water; and we find that $\sqrt{(2ax-x^2)}$=radius of surface of the water, and as the amount of water=one half the cylinder having area of surface of water for base and height of water for altitude, it is equal to $\frac{1}{2}\pi x^2(2a-x)=(4\pi na^3)/3$, (using n instead of $1/m$). Reducing we have $x^3-2ax^2=(-8na^3)/3$. This being a cubic equation, it has three roots; and as their product is negative, one, or all of them must be negative; but as their sum is positive at least one of them must be positive; and taking both of them together, there must be two positive roots, and therefore x has two values, each of which answers the conditions. From our equation we have $n=3(2ax^2-x^3)/8a^3$. Differentiating and reducing we find $x=4a/3$, and substituting this in the expression for the value of n, we have $n=1/m=\frac{4}{9}$. When $1/m=\frac{4}{9}$, $x=4a/3$, and the equation has two equal positive roots—*practically* but not *mathematically*, an exception to the proposition that x has two values, each of which satisfies the conditions.

Also solved by *H. C. WHITAKER* and *G. B. M. ZERR*.

121. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(x^5+y^5+z^5)^3+(x+y)^2=31$.

$(x^5+y^5+z^5)^3+(x+y+z)^3=729$.

$(x+y)^2+(x+y+z)^3=31$.

Solution by W. F. BUCK, Instructor in the Science Department, Leominster High School, Leominster, Mass.

Let $x^5+y^5+z^5=r$....(4).

$x+y=s$....(5).

$x+y+z=t$....(6).

Then from the original equations,

$$r^3+s^2=31,\quad r^3+t^3=729,\quad s^2+t^3=31,$$

which easily give

$$r=\sqrt[3]{\tfrac{729}{2}},\quad s=\sqrt{-\tfrac{667}{2}},\quad t=\sqrt[3]{\tfrac{729}{2}}\;....(7).$$

From (5) and (6), $z=t-s$. Therefore, from (4), $x^5+y^5=r-(t-s)^5$.

Dividing this by (5) and subtracting (5) raised to fourth power from the result

$$x^2+xy+y^2=\frac{r-(t-s)^5-s^5}{-5sxy}.$$

But from (5), $x^2+xy+y^2=s^2-xy$.

$$\therefore\ x^2y^2-s^2xy=\frac{r-(t-s)^5-s^5}{5s},\ \text{and}\ xy=\tfrac{1}{2}s^2\pm\sqrt{\frac{r-(t-s)^5-s^5}{5s}+\frac{s^4}{4}}\ldots(8).$$

Subtracting (8) multiplied by 4 from (5) squared, and taking square root obtain $x-y$, from which with (5),

$$x=\tfrac{1}{2}s\pm\tfrac{1}{2}\sqrt{-s^2\mp4\sqrt{\frac{r-(t-s)^5}{5s}+\frac{s^4}{4}}}$$

whence x by substitution from (7), then x and y from (5) and (6).

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

122. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A man buys a five per cent. ten-year bond at such a price as enables him to spend annually three per cent. upon his investment and by continually investing the residue of the annual interest and its increase annually at four per cent., at the end of term upon payment of his bond has his original investment. What price per $100 does he pay for the bond?

Solution by D. G. DORRANCE, JR., Camden, Oneida County, N. Y.; and H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.

At the beginning of the time, the man pays $$x$ for the bond; at the end of the time, he receives $100 and an accumulated annuity of ($$5-.03x$) running for 10 years at 4%. The value of the annuity at that time was $25(1.04^{10}-1)(5-.03x),=12.0061(5-.03x)$.

Hence $x=100+12.0061(5-.03x)$, from which $x=\$117.6537$.

By a slightly different construction which allows for only *nine* years' expenditure, Professor Zerr obtains the result $116.548.

MECHANICS.

118. Proposed by M. E. ANDERSON, Minneapolis, Minn.

A closed steel cylinder of length L and diameter D is placed in a horizontal position. The cylinder is filled with water to a depth (a) from the lower side, the space above the water being filled with air at a pressure P_1.

What work will be done against this increasing pressure, and against gravity, by a pump forcing water into this tank until the pressure has increased to P_2? Suppose the level of the water in the tank at the beginning to be the same as that of the reservoir from which the water is pumped.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let v_1=volume of gas at pressure P_1.

v_2=volume of gas at pressure P_2.

v=volume of gas at pressure P.

Work done by isothermic expansion from v_2 to v_1 is

$$w=\int_{v_2}^{v_1} Pdv.$$

But $P=P_2v_2/v$.

$\therefore\ w=P_2v_2\int_{v_2}^{v_1}\frac{dv}{v}=P_2v_2\log(v_1/v_2)$.

The work done against gravity by lifting C cubic feet of water through an average height $\frac{1}{2}l$ is $W=\frac{1}{2}Cl\times62\frac{1}{2}=\frac{125}{4}Cl$.

Volume of water$=\frac{1}{4}\pi aD^2$, $v_1=\frac{1}{4}\pi D^2(L-a)$.

$$v_2=\frac{P_1v_1}{P_2}=\frac{\pi D^2P_1}{4P_2}(L-a).$$

$$C=v_1-v_2=\tfrac{1}{4}\pi D^2(L-a)\left(\frac{P_2-P_1}{P_2}\right).$$

$$l=\frac{(L-a)(P_2-P_1)}{P_2},\qquad \frac{v_1}{v_2}=\frac{P_2}{P_1}.$$

$$\therefore\ w=\tfrac{1}{4}\pi D^2P_1(L-a)\log\left(\frac{P_2}{P_1}\right).$$

$$W=\tfrac{125}{16}\pi D^2(L-a)^2\left(\frac{P_2-P_1}{P_2}\right)^2.$$

Total work done$=w+W$.

Work done against pressure is the same as the work of expansion.

P_1 and P_2 are supposed to be given in pounds.

120. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

I project an elastic particle along a chord c of a smooth fixed circular rim of diameter d. The coefficient of elasticity between the particle and the rim is e, and the particle continually rebounds. Find the length of the chord described after the nth rebound.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let β, β_1, β_2, β_3, ...β_n be the angles the particle makes with the diameter before the first and after the first, second, third, and nth rebound, respectively, x=length of chord ofter the nth rebound.

Then $\cot\beta = c/\sqrt{(d^2-c^2)}$.

$\cot\beta_1 = e\cot\beta = ec/\sqrt{(d^2-c^2)}$.

$\cot\beta_2 = e\cot\beta_1 = e^2c/\sqrt{(d^2-c^2)}$.

. . . .

$\cot\beta_n = e^nc\sqrt{(d^2-c^2)} = x/\sqrt{(d^2-x^2)}$.

$\therefore\ xcde^n/\sqrt{[d^2-c^2(1-e^{2n})]}$.

If $e=1$, $x=c$.

AVERAGE AND PROBABILITY.

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102. Proposed by PROFESSOR CAVALLIN.

A random straight line is determined by two points taken at random within a sphere; find the average velocity acquired by a particle in descending the line. [No. 6742, *Educational Times*. Unsolved.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let (x, y, z), (u, v, w) be the coördinates of the two random points with center of sphere as origin. Let a=radius, $\sqrt{(a^2-x^2)}=y'$, $\sqrt{(a^2-u^2)}=v'$, $\sqrt{(a^2-x^2-y^2)}=z'$, $\sqrt{(a^2-u^2-v^2)}=w'$. The elevation of the one end of the line above the other$=(u-x)$.

Velocity$=\sqrt{[2g(u-x)]}$. The limits of x are $-a$ and a; of u, x and a and doubled for the case when $u<x$; of y, $-y'$ and y'; of z, $-z'$ and z'; of v, $-v'$ and v'; of w, $-w'$ and w'. Then since $(\frac{4}{3}\pi a^3)^2$ is the number of ways the two points can be taken, we get

$$\triangle = \frac{2}{(\frac{4}{3}\pi a^3)^2}\int_{-a}^{a}\int_{x}^{a}\int_{-y'}^{y'}\int_{-z'}^{z'}\int_{-v'}^{v'}\int_{-w'}^{w'}\sqrt{[2g(u-x)]}\,dxdudydzdvdw$$

$$=\frac{9\sqrt{2g}}{8a^6}\int_{-a}^{a}\int_{x}^{a}\sqrt{(u-x)}(a^2-x^2)(a^2-u^2)dxdu$$

$$=\frac{3\sqrt{2g}}{35a^6}\int_{-a}^{a}(5a+2x)(a^2-x^2)(a-x)^{\frac{5}{2}}\,dx$$

Let $x=a\cos\theta$.

$$\therefore\ \triangle=\frac{192\sqrt{ag}}{35}\int_{0}^{\pi}(7-4\sin^2\tfrac{1}{2}\theta)\sin^8\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta d\theta=\frac{256\sqrt{ag}}{273}.$$

103. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr.University, Palo Alto, Cal.

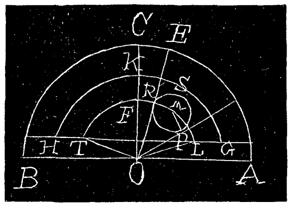
A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a

radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

I. Solution by the PROPOSER.

Let ACB be the given semi-circle, PRS the random circle, O and M their respective centers, OD and OE tangent radii to the circle PSR, OF and CK each equal MP, OC perpendicular to AB, GH parallel to AB, GKH an arc of a circle whose center is O, and LIT an arc of a circle through M and whose center is at O.



Put $MP=OF=CK=x$, $OM=y$, $OA=1$, $\angle ROP=\theta$, arc $LIT=\phi$. Then we have $OK=1-x$, $\theta=2\sin^{-1}(x/y)$, and $\phi=[\pi-2\sin^{-1}(x/y)]$.

Now since the center of the circle PRS may be anywhere in the segment GKH, the limits of y are x and $1-x$; and those of x are 0 and $\frac{1}{2}$. Hence, the required chance is

$$p=\frac{\int_0^{\frac{1}{2}}\int_x^{1-x}\theta\phi\,dx\,dy}{\int_0^{\frac{1}{2}}\int_x^{1-x}\pi\phi\,dx\,dy}$$

$$=\int_0^{\frac{1}{2}}\Bigg[\pi(1-x)^2\sin^{-1}\frac{x}{1-x}-2(1-x)^2\left[\sin^{-1}\frac{x}{1-x}\right]^2$$

$$-4x(1-2x)^{\frac{1}{2}}\sin^{-1}\frac{x}{1-x}+\pi x(1-2x)^{\frac{1}{2}}$$

$$-4x^2\log\left(\frac{1-x}{x}\right)\Bigg]dx\div\pi\int_0^{\frac{1}{2}}\Bigg[\tfrac{1}{2}\pi(1-n)^2$$

$$-(1-x)^2\sin^{-1}\frac{x}{1-x}-x(1-2x)^{\frac{1}{2}}\Bigg]dx=\Bigg[\tfrac{2}{3}(1-x)^3\left(\sin^{-1}\frac{x}{1-x}\right)^2$$

$$-(\pi/3)(1-x)^3\sin^{-1}\frac{x}{1-x}+\left[\tfrac{1}{3}(1-2x)^{\frac{1}{2}}+\tfrac{8}{9}(1-2x)^{\frac{3}{2}}-\tfrac{1}{3}(1-2x)^{\frac{5}{2}}\right]\sin^{-1}\frac{x}{1-x}$$

$$-\frac{\pi}{12}(1-2x)^{\frac{1}{2}}-\frac{2\pi}{9}(1-2x)^{\frac{3}{2}}+\frac{\pi}{12}(1-2x)^{\frac{5}{2}}$$

$$-\tfrac{4}{9}x+\tfrac{4}{9}\log(1-x)-\tfrac{4}{3}x^3\log\left(\frac{x}{1-x}\right)\Bigg]_0^{\frac{1}{2}}$$

$$\div\pi\Big[\tfrac{1}{3}(1-x)^3\sin^{-1}\frac{x}{1-x}-\frac{\pi}{6}(1-x)^3$$

$$+\tfrac{1}{12}(1-2x)^{\frac{1}{2}}+\tfrac{2}{9}(1-2x)^{\frac{3}{2}}-\tfrac{1}{12}(1-2x)^{\frac{5}{2}}\Big]_0^{\frac{1}{2}}$$

$$=\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

MISCELLANEOUS.

Further Remark on Problem 90.

The results of the problem may be put in a better form as follows: e must be a function of s, say $e=f(s)$. For a *continuous* x,

$$f(s+x)=f(s)+x\Delta f(s)+\frac{x(x-1)}{2\,!}\Delta^2 f(s)+\frac{x(x-1)(x-2)}{3\,!}\Delta^3 f(s).$$

Put $s=0$, and substitute for $f(0)$, $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0)$, their values 21, $\frac{7}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, respectively, and

$$f(x)=21+\tfrac{41}{12}x+\frac{x^3}{12};$$

replacing x by a *continuous* s,

$$e=f(s)=21+\frac{41s}{12}+\frac{s^3}{12},$$

which determines once for all the functional relation between any value of s and e. This result is somewhat analagous to the primitive functional relation found from a differential equation, only here difference coefficients enter instead of differential coefficients, and the work is infinitely simpler. Of course the same results could have been obtained by La Grange's formula of interpolation. In order to use the above formula for calculating e for any distance s, 100 yards must be taken as the unit. E. D. Roe, Jr.

93. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

Prove that $-(\sqrt{-1})^{\sqrt{-1}}=e^{(\sqrt{-1}-\frac{1}{2})\pi}$.

I. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.**

In the Napierian system of logarithms we always have $a^n=e^{n\log a}$....(1).

Put $\sqrt{-1}=i$; then $-1=i^2$, and $-i^i=i^2.i^i=i^{i+2}$....(2).

In (1), putting $a=i$, $n=i+2$, gives $i^{i+2}=e^{(i+2)\log i}$ ....(3).

But $\log i=(2n\pi+\frac{1}{2}\pi)i$....(4); then (3) is, with $n=0$,

$$i^{i+2}=e^{(i+2)(\frac{1}{2}\pi)i}=e^{(i-\frac{1}{2})\pi}.$$

II. Solution by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y.; GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, State University, Eugene, Ore.; and HARRY S. VANDIVER, Bala, Pa.

Since $\cos\theta+i\sin\theta=e^{i\theta}$ $[i=\sqrt{-1}]$ we have $i=e^{i\frac{1}{2}\pi}$, $i^i=e^{-\frac{1}{2}\pi}$, $-i^i=e^{i\pi}.e^{-\frac{1}{2}\pi}$.

$$\text{or } -(\sqrt{-1})^{\sqrt{-1}}=e^{(\sqrt{-1}-\frac{1}{2})\pi}.$$

III. Solution by CHARLES PURYEAR, Department of Mathematics, Agricultural and Mechanical College, College Station, Texas.

$e^x=1+x+\dfrac{x^2}{2\,!}+\dfrac{x^3}{3\,!}+\dfrac{x^4}{4\,!}+\ldots.(1)$.

$e^{-x}=1-x+\dfrac{x^2}{2\,!}-\dfrac{x^3}{3\,!}+\dfrac{x^4}{4\,!}-\ldots.(2)$.

$\therefore\ e^x+e^{-x}=2(1+\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}+\ldots.)\ldots.(3)$.

Replacing x by ix where $i=\sqrt{-1}$,

$e^{ix}+e^{-ix}=2(1-\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}+\ldots.)\ldots.(4)$, or $e^{ix}+e^{-ix}=2\cos x\ldots.(5)$.

Let $x=\pi$, then $e^{\pi i}+e^{-\pi i}=-2\ldots.(6)$.

Solving, $e^{\pi i}=-1\ldots.(7)$.

Extracting the square root of each member of (7), $e^{(\frac{1}{2}\pi)i}=i\ldots.(8)$.

Raising each member of (8) to the power of i, $e^{-\frac{1}{2}\pi}=(i)^i$ ...(9).

Multiplying (7) and (9), $e^{(i-\frac{1}{2})\pi}=-(i)^i$.

Also solved by *J. SCHEFFER*, *H. C. WHITAKER*, and *G. B. M. ZERR*.

94. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The wall of a house, if its plane were extended, would cut the horison at an angle$=\beta^\circ$ south of the true east point. The latitude of the place being$=\phi$, and the declination of the sun$=\delta$. When will the sun cease to shine through a window in that wall?

Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

When the sun does not shine in the window its azimuth is $270^\circ+\beta$. Let P be the pole, Z the zenith, and S the sun; then $PZ=co-\phi$, $PS=co-\delta$, angle $Z=90^\circ+\beta$.

Therefore, by spherical trigonometry, $\sec\phi\tan\beta\sin P-\tan\phi\cos P=-\delta$.

$$\sin P = \frac{-\sec\phi\tan\beta\tan\delta \pm \tan\phi(\sec^2\phi\tan^2\beta + \tan^2\phi - \tan^2\delta)^{\frac{1}{2}}}{\sec^2\phi\tan^2\beta + \tan^2\phi}.$$

The time A. M. $=12-P^\circ/15^\circ$.

Also solved by *G. B. M. ZERR*, and *J. SCHEFFER*.

95. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

"I enjoy here," said Goethe, "both good days and good nights. Often before dawn I am already awake, and lie down by the open window to enjoy the splendor of the three planets, which are at present to be seen together, and to refresh myself with the increasing brilliancy of the morning red," This was written in the summer of 1828 near Weimar. See Goethe's "Conversations with Eckermann," Bohn's Library, 1898, page 323.

What three planets are referred to ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

We will only consider Mercury, Venus, Mars, Jupiter, and Saturn, as Uranus and Neptune are too faint to reveal any splendor. The event would happen soon after conjunction.

Mars and Sun were in conjunction January 16, 1900.

Jupiter and Sun were in conjunction November 13, 1899.

Saturn and Sun were in conjunction December 29, 1900.

Venus and Sun were in conjunction inferior July 4, 1900.

Mercury and Sun were in conjunction superior February 9, 1900.

Synodic period of Mars, 780 days ; of Jupiter, 399 days ; of Saturn, 378 days ; of Venus, 584 days ; of Mercury, 116 days.

From July 1, 1828, to January 16, 1900, are 26132 days. $26132\div780=33$ and 392 days over. Therefore the conjunction of Mars and the Sun happened 392 days after July 1, 1828, and so Mars was not one of the three planets.

From July 1, 1828, to November 13, 1899, are 26068 days. $26068\div399=65$ and 133 days over. Therefore Jupiter could not have been one of the three planets.

From July 1, 1828, to December 29, 1900, are 26479 days. $26479\div378=70$ and 19 days over. Therefore the conjunction of Saturn and the Sun happened only 19 days after July 1. 1828.

From July 1, 1828, to July 4, 1900, are 26301. $26301\div584=45$ and 21 days over. Therefore the conjunction of Venus (inferior conjunction) happened only 21 days after July 1, 1828.

From July 1, 1828, to February 9, 1900, are 26156 days. $26156\div116=225$ ans 56 days over. Therefore superior conjunction of Mercury and the Sun happened 56 days after July 1, 1828. Thirty-six days days previous to this or 20 days after July 1, 1828, Mercury was at greater elongation and therefore nearly as bright as Sirius.

Therefore Mercury, Venus and Saturn are the three planets referred to and the time the latter part of July.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

146. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance. O.

If the driving-wheels of Locomotive No. 200 on the Pennsylvania Railroad, $m=7$ feet in diameter, turn $n=20$ times in $p=3$ seconds, and lose $r=12\%$ of their forward motion by slipping on the smooth steel rails, at what rate per hour is the locomotive moving over the rails?

147. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defianca, O.

Stock bought $m=10\%$ above par pays $p=8\%$ on the investment. What per cent. will it pay if bought at $n=10\%$ discount?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Nov 10.

ALGEBRA.

142. Proposed by A. H. BELL, Hillsboro, Ill.

If x/y is the convergent preceding the complete quotient $(\sqrt{A}+m)/n$; prove that $x^2-Ay^2=\pm n$.

143. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $x+y+z+u=a\ldots.(1).$

$$x^2+y^2+z^2+u^2=b\ldots.(2).$$

$$x^3+y^3+z^3+u^3=c\ldots.(3).$$

$$x^4+y^4+z^4+u^4=d\ldots.(4).$$

144. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Show that the number of ways in which 15 different problems may be distributed among 5 students so that each student shall have three of them, is $N=(5.3)!/(3!)$.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

GEOMETRY.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette. Stroud, Gloucestershire, England.

The center N of the 9-point circle of a triangle ABC lies on P, the pedal line of a point on the circumcircle. Find the angle of intersection of P and AB.

173. Proposed by P. C. CULLEN, Principal of Schools, Indianola, Neb.

To construct circle tangent to a given line at a given point such that tangents drawn to this circle and passing through two fixed points shall be parallel.

174. Proposed by J. M. HOWIE, Professor of Mathematics, The Nebraska State Normal School, Peru, Neb.

Describe a circle which shall pass through a given point and be tangent to two given circles.

*** Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

CALCULUS.

135. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

To find the equation of the evolute of the common catenary

$$y=(\tfrac{1}{2}c)(e^{c/x}+e^{-c/x}).$$

136. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Evaluate the definite integral

$$\int_0^1\int_0^1 \frac{v^{l-1}u^{m-1}(1-v^n)^{p-1}(1-u^s)^{r-1}dvdu}{[bv^n+c(1-v^n)]^{p+l/n}(u^s+a)^{r+m/s}}.$$

137. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Develop the equation of the curve assumed by the inextensible and revolving skipping rope.

*** Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

MECHANICS.

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A pendulum-bob, weight$=w$, is suspended by a perfectly elastic cord, length l. This pendulum makes n vibrations *up and down*, through a space of $2m$ inches while it makes a complete vibration in an arc of 2ψ. Determine the nature of the curve described by the center of the pendulum-bob in making one complete vibration in arc.

125. Proposed by THOMAS U. TAYLOR, C. E., Professor of Civil Engineering, University of Texas, Austin, Texas.

(1) If a parabola is described on the verticle face of a reservoir wall, axis vertical and in the surface, and $P\,(h,\,b)$ be any point on the curve, and B the foot of the perpendicular from P on the axis, find c. p. on area OBP.

(2) If A is point where horizontal through P cuts vertical axis (OY), find c. p. on area OAP.

*** Solutions of these problem should be sent to B. F. Finkel not later than Nov. 10.

DIOPHANTINE ANALYSIS.

89. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Show that in $2x^2+2y^2-z^2=\square \ldots.(1)$,

$$2x^2+2z^2-y^2=\square \ldots.(2),$$

$$2y^2+2z^2-x^2=\square \ldots.(3),$$

any two numbers and their sum and difference will satisfy the conditions.

90. Proposed by H. S. VANDIVER, Bala, Penn.

Prove that it is always possible to find an infinite number of positive integral values of x, y and z, such that the relation $z^2=x^2+bxy+cy^2$ is satisfied, b and c being any integers whatever.

*** Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

AVERAGE AND PROBABILITY.

113. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance. O.

A given cube is cut by a plane in such a manner that the *lines of section* form a *regular* hexagon. What is the mean area of this hexagon ?

114. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr.University, Palo Alto, Cal.

If a regular polygon of n sides be placed at random on another equal polygon, show that the chance that the center of the first will fall on the second polygon is $\dfrac{\pi}{2[\pi+n\tan(\pi/n)]}$.

*** Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

MISCELLANEOUS.

114. Proposed by J. SCHEFFER. A. M.. Hagerstown, Md.

When the sun's declination was 15° N. his altitude was found to be 20°, and after an hour's interval his altitude was found to be 31°. Required the latitude of the place of observation.

115. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Determine geometrically where to stand so as to be able to throw a stone over a tree with the *minimum* velocity.

116. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

Prove that

$$-\,|\alpha_1\beta_2\gamma_3|^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2=\begin{vmatrix}
\begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \alpha_1 \\ b & d & e & \alpha_2 \\ c & e & f & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\
\begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \beta_1 \\ b & d & e & \beta_2 \\ c & e & f & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix} \\
\begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 & 0 \end{vmatrix} &
\begin{vmatrix} a & b & c & \gamma_1 \\ b & d & e & \gamma_2 \\ c & e & f & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{vmatrix}
\end{vmatrix}.$$

[From *Muir's Determinants*].

*** Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

NOTES.

Professor T. H. Stafford, of Williams College, died June 13th, aged sixty-seven years.

Professor C. S. James, formerly Professor of Mathematics and Physics in Bucknell College, died June 8th.

In the October number of THE MONTHLY will appear a paper "On Systems of Isothermal Curves," by Dr. L. E. Dickson.

Professor L. T. Neikirk, Boulder, Colorado, has been given a Fellowship in Mathematics in the University of Pennsylvania and has gone there for research work.

Dr. L. E. Dickson has in press a College Algebra. This work is being published by John Wiley & Sons and its appearance will be looked forward to with much interest by both mathematicians and teachers of mathematics.

Dr. G. A. Miller, formerly of Cornell University, but now of Leland Stanford University, will give the following courses the present academic year: *Trigonometry*, 2 hours; *History of Mathematics*, 2 hours; *Theory of Numbers*, 2 hours; *Group Theory*, 2 hours.

Professor F. P. Matz has been elected Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio. In the October number of THE MONTHLY will appear a brief biography of Dr. Thomas Craig, which biography has been prepared by Professor Matz.

BOOKS.

Higher Algebra. By George Egbert Fisher, A. M., Ph. D., and I. J. Schwatt, Ph. D., Assistant Professors of Mathematics in the University of Pennsylvania. 8vo, Cloth sides and Leather back. xviii+615+xviii pages. Published by the authors.

This book is such an arrangement of the authors' *Elements of Algebra* and *School Algebra*, together with such revisions and additions of new matter as to make a course of study in the subject suitable for use in the first year in colleges. To the end of Chapter XXVIII the *Higher Algebra* is identical with the *Elements of Algebra* and the *School Algebra*. These two books have been most favorably mentioned in previous issues of this Journal. In the additions incorporated in the Higher Algebra, the authors have maintained the same felicity of expression, simplicity and rigor in demonstration as has characterized their previous works. They have, in the Higher Algebra, prepared a text-book which will be highly appreciated by all who use it. B. F. F.

Theoretical Mechanics. An Elementary Treatise. By W. Woolsey Johnson, Professor of Mathematics, U. S. Naval Academy. 12mo, cloth, xv+434 pages. Price, $3.00. New York: John Wiley & Sons.

In this book the author has taken great care in laying the fundamental principles of the subject on a firm foundation. No formal divisions of the subject into Kinematics, Statics, and Kinetics has been made. The topics usually included under the first head are introduced separately, each at the point where it is required for immediate application to the treatment of the motions produced by forces. In like manner, the other subjects have received such treatment as seemed to the author the most logical method. Special prominence is given to those results which it is the most important to make familiar to the student of Applied Mechanics. Numerous interesting problems have been selected as exercises coming under the various subjects. B. F. F.

One Hundred Problems in Mathematical Physics. By E. P. Thompson, A. M., Professor of Mathematies, Miami University, Oxford, Ohio. 8vo, cloth, 52 pager. Published by the author.

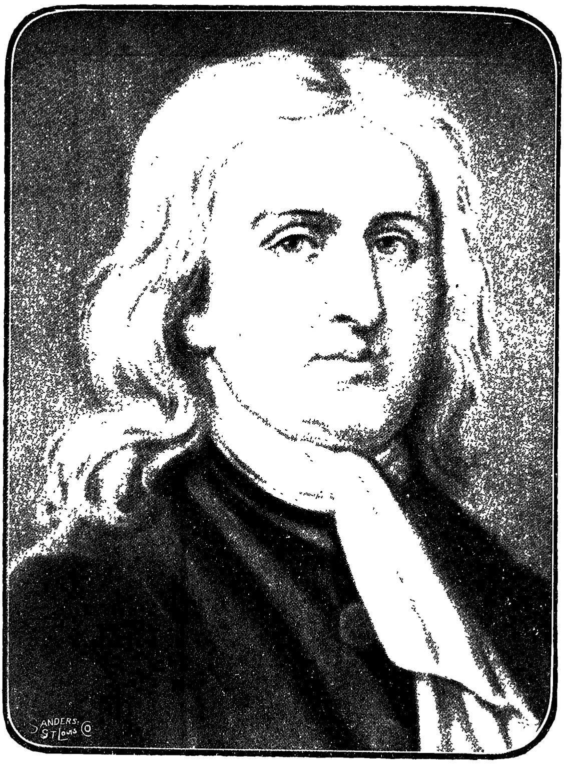
This little book contains interesting problems on the steam-engine, the dynamo, the top, gigroscope and bicycle; and on the use of instruments, graphics, quaternions, least squares, strength of materials, gunnery, and other subjects. The book is intended to be used in connection with mathematical and physical classes and will be found helpful and suggestive to teachers and students. B. F. F.

Original Investigation or How to Attack an Exercise in Geometry, With many Model Solutions and a Complete Discussion of the Principles Underlying the Same. By Elisha S. Loomis, Ph. D., Head of Mathematical Department of the West High School, Cleveland, Ohio. 8vo, Flexible cloth. vi+62 pages. Boston and Chicago. Ginn & Co.

In this little work is found many suggestions of the highest value to teachers. In it, Professor Loomis has given the results of his twenty years' experience in teaching geometry and these will be most helpful to all teachers desiring to present the subject to to their pupils in most pedogogical and scientific manner. We most heartily recommend it to the study of both teacher and pupil. B. F. F.

College Algebra. By James Harrington Boyd, Ph. D., Assistant Professor of Mathematics in the University of Chicago. 8vo, cloth and leather back. xxi+787 pages. Price, $2 00. Chicago: Scott, Foresman & Co.

This is one of the most,if not the most voluminous algebras that has yet been published in this country. Its size having increased its cost, together with the fact that it is too comprehensive to be covered in the time alloted to the subject in the first year of the college course, will hinder to some extent the adoption of this book in some schools. However, where the students can afford to purchase a good book, it is best to put into their hands the best possible texts so that they may gain a knowledge of those refined generalizations and demonstrations which have come into fairly general use at the present time. This book has been carefully prepared. Special attention is given to rigor and logical sequence which is demanded by the best teachers, to the development of the number concept, and to the use of geometrical illustrations in so far as they very strikingly force the attention of the student to the principle involved. Emphasis is placed upon the principle of the permanence of form, the discussion of the irrational, the theory of fractional exponents, and complex numbers. The book is well written and is worthy of extended use in colleges and universities. B. F. F.



ISAAC NEWTON.

THOMAS CRAIG.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. OCTOBER, 1901. No. 10.

BIOGRAPHY.

PROFESSOR THOMAS CRAIG, C. E., PH. D.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronoy in Defiance College, Defiance, Ohio.

The subject of this sketch was born at Pittston, Pennsylvania, December 20, 1855. He was the son of Alexander Craig, a Scotch mining engineer. In the common schools of his native town he distinguished himself for his assiduity and success in study. From the very beginning of his scholastic career, he was a remarkably bright student. With several of his townsmen, he entered Lafayette College in September, 1871. Four years later he was graduated with the degree of Civil Engineer, and delivered the "Scientific" *Honorary* Oration. The same year he received honorable mention in mathematics at the Inter-Collegiate Contest in New York City. While a student in Lafayette College, he was a member of the Franklin Literary Society, and of the *Upsilon Beta* and of the *Delta Kappa Epsilon* Fraternities.

Says Registrar Coffin of Lafayette College: "In Pittston he received enthusiasm from Prof. William J. Bruce, a teacher who made him his protege. His conduct was always exemplary, and his moral character unspotted. For digging up a lamp post belonging to the city of Easton and transplanting it to the grounds of Lafayette College, he and some other students were detected by the police, fined by the Mayor, and coercively removed (rusticated) by the Faculty to Frazer, Pennsylvania, to study alone for one month under the charge of that accomplished teacher, Rev. John C. Clyde, D. D., pastor of the Presbyter-

ian Church. This method of treatment cured the entire party of a desire to engage in further mischief."

After receiving the degree of Civil Engineer from Lafayette College, he engaged in studying mathematics and teaching at Newton, New Jersey. As a student of advanced mathematics under the direction of Professor Sylvester, he entered the Johns Hopkins University in 1876. He was one of the first persons elected to a fellowship in mathematics at the Johns Hopkins University. This fellowship he held from 1876 to 1879, and received the degree of Doctor of Philosophy in 1878. He began lecturing in this University when he was a student. After receiving the degree of Doctor of Philosophy, he became connected with the United States Coast and Geodetic Survey, for which he prepared in 1879 a Treatise on the Mathematical Theory of Projections. For many years he was associated with *The American Journal of Mathematics* as a contributor and as assistant editor; and from 1894 to 1899, he was the responsible editor of this mathematical journal.

He rose, by successive promotions, from the grade of Fellow to those of Associate, Associate Professor, and Professor of Mathematics. During his stay in Washington he studied the Theory of Functions from the work of Königsberger, under the guidance of Professor Simon Newcomb. Among his special studies may be mentioned the theory of functions, differential equations, mechanics, and hydrodynamics. He kept pace with the most rapid advances in mathematical learning. Many of his contributions can be found in the *American Journal of Mathematics* from 1876 to 1900, and many other of his contributions have been published in various foreign mathematical journals. He was a member of a number of mathematical and scientific societies.

With subjects which received extensive development in the hands of such illustrious mathematicians as Sylvester, Cayley, Hermite, Poincaré, Darboux, Appel, Picard, Abel, and others, he was actively engaged for the benefit of the readers of the *American Journal of Mathematics*. In the capacity of editor of this mathematical journal, Dr. Craig was eminently successful in securing the contributions of distinguished mathematicians of England and the Continent, whose letters show high appreciation of the abilities of their American editor and correspondent.

That Dr. Craig was very *optimistic* is well known to the writer, whose senior he was by some years. Writes President Gilman in his Annual Report of the University of 1900: "Kindness toward young men and readiness to encourage them were among his admirable qualities." Speaking from a personal acquaintance extending over many years, the writer always found Dr. Craig to be an efficient workman, a pleasant companion, a warm friend, and a good man. With his students he associated familiarly, and occasionally he invited them to spend a mathematical evening at his home. Dr. Craig was an admirable lecturer. He had the ability to communicate what is known. His lectures were always thoroughly prepared; and he always had a comprehensive, accurate, and clear knowledge of what he intended to impart. It is well known that, as a lecturer,

Dr. Craig thought quickly, spoke rapidly, and wrote with great celerity. It is true that some of his lectures on differential equations, on hydrodynamics, and on the theory of functions were very advanced and very difficult; and those lectures could be followed with profit only by the maturest of his students.

There is published by the D. Van Nostrand Company his *On the Motion of a Solid in a Fluid.* His most elaborate work *Linear Differential Equations* is published by John Wiley & Sons.

In a private letter, Professor Coffin of Lafayette College says: "His was a wonderfully strong mind."

Likewise, Prof. Simon Newcomb says: "As a student Dr. Craig had a remarkable power of rapid comprehension of mathematical treatises."

Likewise, President Gilman says: "Dr. Craig was a man of unusual aptitudes and of great promise, who began a brilliant career with us; but he suffered long from ill health—and of late years, he was the victim of insomnia."

"Likewise, his father—Mr. Alexander Craig, says: "My son, Thomas Craig, was married to Miss Louise Alvord, of Washington, D. C., May 4, 1880. She is the daughter of the late General Benjamin Alvord of the United States Army"

A Baltimore paper says: "Dr. Thomas Craig, the Professor of Pure Mathematics in the Johns Hopkins University, died suddenly of heart failure, May 8, 1900, at his residence, 1822 St. Paul Street. He had been complaining of feeling unwell for several weeks, although he was at the University as usual that morning. He returned home about noon, and went to his room for a rest before dinner. A member of the family went to call him for dinner and found him dead. Coroner Saunders gave a verdict of death from heart failure."

Dr. Craig is survived by his father and a sister, Miss Margaret Craig, both of Pittston, Pennsylvania; and, also, by his widow and two daughters, Miss Alisa Craig and Miss Ethel Craig, who reside in Baltimore. Says his father: "He was buried at Pittson, Pennsylvania, by the side of his mother whom he always loved so well."

In his Annual Report of the University for 1900 President Gilman writes: "The death of Professor Craig occurred on the 8th of May, 1900, after a long period of declining powers. Those who knew Dr. Craig only in his declining years, need to be told of the enthusiasm, the diligence, and the learning which for a long period were his distinguishing characteristics. He was one of a company of bright young mathematicians who came to the University in its first year, attracted by the brilliant reputation of Professor Sylvester. He showed at once extraordinary powers of acquisition, as well as great ability in the treatment of certain subjects in the domain of higher mathematics. In addition to his contributions to mathematical journals, he published, in 1879, two manuals on the elements of the mathematical theory of fluid motion, and, in 1889, the first volume of a treatise on linear differential equations, a continuation of which was not completed at the time of his death."

In the *American Journal of Mathematics* for October, 1900, Prof. Simon

Newcomb writes as follows: "Thomas Craig, the former editor of this journal, and Professor of Pure Mathematics in the Johns Hopkins University, died May 8, 1900, in his forty-fifth year. His connection with the *American Journal of Mathematics*, as editor or associate editor, continued through the greater part of its existence, being severed at the end of 1898, when failing health compelled him to retire from the editorship. Craig was connected with the Johns Hopkins University from its foundation. He was attracted thither by the desire to pursue mathematical studies under the guidance of Sylvester. From the beginning he showed an extraordinary development of the faculty of acquisition, being able to master, almost without effort, the writings of any of the great geometers to which he was attracted. The productive faculty was developed more slowly.

He was naturally among the earliest Doctors of the University, and the first, or one of the first, to graduate in mathematics. His earliest publications were two small books on hydrodynamics, and a work on projections, prepared for the U. S. Coast Survey, with which he was associated for a short period after his graduation. His most elaborate separate work was a treatise on Linear Differential Equations, embodying the course of instruction on that subject which he gave to the students of the University. A work on higher geometry, on which he was engaged, but, so far as the writer is aware, on which he had made little progress, were left unfinished at the time of his death.

He was also a frequent contributor to the pages of this journal. Among the contributions worthy of especial mention were his various papers on Theta functions, in the fifth and sixth volumes, and a memoir on Linear Differential Equations whose fundamental integrals are the successive derivatives of the same function, in the eighth volume.

During his editorship he devoted himself with great energy to the interests of the *Journal*. The principal object of at least one of his visits abroad was to interest European geometers in it. He recognized and admired the genius of Poincaré; and two elaborate memoirs by the latter, which appeared in the seventh and eighth volumes, were believed to have been sent to the *Journal* on Craig's personal solicitation.

As an expounder of mathematical subjects to advanced students, Craig's abilities were of a high order. His lectures were well prepared, and he spoke with rapidity, clearness and force. It may well be that only the best students were able to keep up with him, but these profited in a high degree from his expositions and entertained a permanent appreciation of his efforts for their development. Concentrating his interests almost entirely on his family and his students, rarely taking a long rest, he mingled little with men, especially in his later years, when his activities were greatly restricted by failing health."

At a meeting of the (Johns Hopkins) Board of University Studies, held May 23, 1900, the following minute was unanimously adopted:

"The members of the Board of University Studies of the Johns Hopkins University desire to express their sorrow at the death of their friend and colleague, Prof. Thomas Craig, who, as student and teacher of mathematics, had

been connected with the University for nearly the entire period of its existence. One of the brilliant young men whom Professor Sylvester attracted to the University in its early days, he won straightway the favorable notice of that eminent man for the enthusiasm and intellectual acumen with which he entered upon the study of advanced mathematics, then almost an unknown science in this country; and this fortunate combination of interest, energy, and ability characterized his entire career. At the time of his death he was occupied in the preparation of a treatise on the Theory of Surfaces. Undoubtedly the intense ardor with which he engaged in this work contributed in large measure to that impairment of the nervous system from which he had recently suffered. Professor Craig possessed great power of research, and wrote much for various mathematical journals. For many years he was editor of the *American Journal of Mathematics*, and it is largely due to his zeal and able direction that that journal continues to hold its high rank in the mathematical world. Professor Craig occupied a place in the very front rank of American mathematicians. His scientific ideals were the highest, and as teacher, editor, and investigator, he brought to his work a high degree of originality, and an intellectual ardor which was a source of inspiration to all with whom he was closely associated."

ON SYSTEMS OF ISOTHERMAL CURVES.*

By PROFESSOR L. E. DICKSON.

1. The object of this paper is to give an elementary geometrical definition of a system of isothermal curves in the plane. The definition is readily extended to families of curves on any algebraic surface. For simplicity of expression, the definition is given in connection with the two families of curves which are to be discussed at length; the general definition will then be apparent. The usual methods of treating the subject are indicated in §§ 4-5.

2. The concentric circles about a point O have as orthogonal trajectories the straight lines through O. Select two of these lines, OP and OQ, and designate by q the number of radians in the angle POQ. On the circle about O with radius $OP=r$, the arc $PQ=qr$. On the line OP measure off from P the length $PT=qr$. [In Fig. 1, it is taken to the right of P]. On the circle about O with radius $OT=r+qr$, the arc TS equals $qr+q^2r$. Hence PQ, PT and QS are each of length qr, while the limit of $TS \div PT$ as q approaches zero is unity.

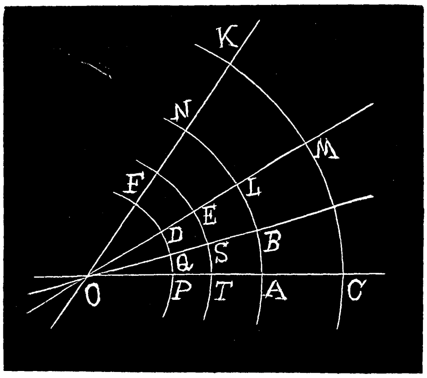


Fig. 1.

*Read before the American Association for the Advancement of Science.

Let L be an arbitrary point in the plane, but distinct from O. Denote by A and B the intersections of OP and OQ, respectively, with the circle about O with radius OL. Denote by D and E the intersections of OL with the circles through P and T. We define a curvilinear quadrilateral, one of whose vertices is L, as follows: Set $OA=a$, and denote by d the number of radians in the angle POD. Then arc $AB=qa$, $DE=qr$. On OP take $AC=AB=qa$; on the circle PQ take arc $DF=DE=qr$. The lines OD, OF and the circles about O with radii OA, OC intersect in the curvilinear quadrilateral $LNKM$. Its sides have the following lengths:

$$LM=qa, \quad LN=qa, \quad NK=qa, \quad MK=q(a+qa).$$

Hence, as q approaches zero, the limits of the ratios of the sides are all unity. The angles of the quadrilateral are always $\pi/2$. In the vicinity of an arbitrarily chosen point L, we obtain by this construction a network of rectangular curvilinear quadrilaterals, which tend to become squares as q approaches zero. The family of concentric circles and their orthogonal trajectories are therefore said to form a system of isothermal curves.

Instead of the particular "base lines" OP, OQ, circle PQ, and circle TS, we may employ any other pair of straight lines through O and pair of circles about O, such that the limits of the ratios of the quadrilateral formed are all unity.

3. Consider the family of circles $x^2+(y-r)^2=r^2$ tangent to the x-axis at the origin. In polar coördinates, their equations are

$$(1) \qquad \rho=2r\sin\theta \quad (r=\text{parameter}).$$

Their orthogonal trajectories are the circles tangent to the y-axis at the origin; their equations are

$$(2) \quad \rho=2R\cos\theta \ (R=\text{parameter}).$$

We are to show that these curves form an isothermal system, according to the definition next given. Employing as base lines certain of the circles, PQ, TS, PT, and QS, we define a curvilinear quadrilateral $LNKM$, one of whose vertices L is an arbitrarily chosen point distinct from O, by making arc $AB=$arc AC, arc $DF=$arc DE, thereby determining the circles $OCMK$ and $OFNK$. If $PT=PQ$, then $LNKM$ tends to become a square as PQ approaches zero.

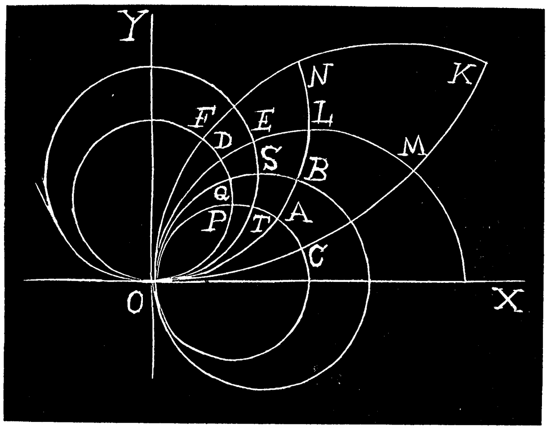


Fig. 2.

In proof, we consider the circles of unit radius

$$OPQ : \rho=2\sin\theta\,; \quad OPT : \rho=2\cos\theta.$$

Their second point of intersection P has the coördinates

$$P : (\rho,\ \theta)\equiv(\sqrt{2},\ \pi/4).$$

Let the point Q be determined on the circle OPQ so that the angle POQ shall contain q radians. Then arc $PQ=2q$. Take arc $PT=2q$ on the circle OPT. Since angle $XOQ=\frac{1}{4}\pi+q$, the length of OQ is

$$\rho=2\sin(\tfrac{1}{4}\pi+q)=\sqrt{2}(\cos q+\sin q)=\sqrt{2}(1-\frac{q^2}{2\,!}+\frac{q^4}{4\,!}-\ldots.+q-\frac{q^3}{3\,!}+\frac{q^5}{5\,!}-\ldots).$$

Hence the coördinates of Q are

$$Q : \rho=\sqrt{2}(1+q+\ldots.).\quad \theta=\tfrac{1}{4}\pi+q,$$

where, as henceforth, the terms indicated by dots contain the factor q^2. By a like proof, or by symmetry, the coördinates of T are

$$T : \rho=\sqrt{2}(1+q+\ \ldots),\quad \theta=\tfrac{1}{4}\pi-q.$$

In order that a circle (2) shall contain Q, we find that

$$R=\frac{\sqrt{2}(1+q+\ldots.)}{2\cos(\frac{1}{4}\pi+q)}=\frac{1+q+\ldots.}{1-q+\ldots.}=1+2q+\ldots.$$

Hence the equation of the circle in question is

$$OQSB : \rho=2\cos\theta(1+2q+\ldots\ ).$$

Analogously, or by symmetry, the circle (1) through T is

$$OTSE : \rho=2\sin\theta(1+2q+\ldots.).$$

Consider an arbitrary point L. It may be determined as the intersection of the circle $OABL$ of set (1) with the circle $ODEL$ of set (2). These circles are determined by the points A and D, respectively. Let them have the coördinates

$$A : (\rho,\ \theta)\equiv(2\cos\alpha,\ \alpha)\,; \quad D : (\rho,\ \theta)\equiv(2\sin\beta,\ \beta).$$

Then the circles $OABL$ and $ODEL$ have the respective equations

$$\rho=2\cot\alpha.\sin\theta,\quad \rho=2\tan\beta.\cos\theta.$$

The former intersects the circle $OQSB$ in the point B for which

$$\tan\theta_B = \tan\alpha.(1+2q+\ldots\ldots).$$

Denoting by σ the angle AOB, we have $\theta_B = \alpha+\sigma$, so that

$$\tan\alpha.(1+2q+\ldots\ldots)=\frac{\tan\alpha+\sigma+\frac{1}{3}\sigma^3+\ldots\ldots}{1-\tan\alpha(\sigma+\frac{1}{3}\sigma^3+\ldots\ldots)}=\tan\alpha+\sigma+\sigma\tan^2\alpha+\sigma^2(\ ).$$

Hence $\sigma=q\sin2\alpha+\ldots\ldots$ Since the radius of $OABL$ is $\cot\alpha$, we get

$$\mathrm{arc}AC\equiv\mathrm{arc}AB=2\sigma\cot\alpha=4q\cos^2\alpha+\ldots\ldots$$

Hence the point C has the coördinates

$$\rho=2\cos\alpha+4q\sin\alpha\cos^2\alpha+\ldots\ldots,\quad \theta=\alpha-2q\cos^2\alpha+\ldots\ldots$$

In order that circle a (1) shall pass through C, we must have

$$r=\frac{\cos\alpha+2q\sin\alpha\cos^2\alpha+\ldots\ldots}{\sin(\alpha-2q\cos^2\alpha+\ldots\ldots)}=\frac{\cos\alpha(1+2q\sin\alpha\cos\alpha+\ \ldots)}{\sin\alpha-2q\cos^3\alpha+\ldots\ldots}$$

$$=\frac{\cos\alpha}{\sin\alpha}[1+2q(\sin\alpha\cos\alpha+\frac{\cos^3\alpha}{\sin\alpha})+\ldots\ldots]=\cot\alpha(1+2q\cot\alpha+\ldots\ldots).$$

The equation to the circle $OCMK$ is therefore

$$\rho=2\sin\theta(\cot\alpha+2q\cot^2\alpha+\ldots\ldots).$$

For the intersection E of the circles $ODEL$ and $OTSE$, we have

$$\tan\theta_E=\tan\beta.(1-2q+\ldots\ldots),\quad \theta_E=\beta-q\sin2\beta+\ldots$$

$\therefore$ $\mathrm{arc}DF\equiv\mathrm{arc}DE=4q\sin^2\beta+\ \ldots$
Hence the point F has the coördinates

$$\rho=2\sin\beta+4q\cos\beta\sin^2\beta+\ldots\ldots,\quad \theta=\beta+2q\sin^2\beta+\ldots\ldots$$

The equation to the circle $OFNK$ of family (2) is therefore

$$\rho=2\cos\theta(\tan\beta+2q\tan^2\beta+\ldots\ldots).$$

We may now determine the coördinates of L, M, N from the equations of the circles $OABL$, $ODEL$, $OCMK$, and $OFNK$:

$$\tan\theta_L=\tan\alpha\tan\beta,\quad \rho_L=2(\tan^2\alpha+\cot^2\beta)^{-\frac{1}{2}},$$

$$\tan\theta_M = \tan\alpha\tan\beta - 2q\tan\beta + \ldots,$$
$$\tan\theta_N = \tan\alpha\tan\beta + 2q\tan\alpha\tan^2\beta + \ldots.$$

Denoting by τ the angle MOL, then $\theta_M = \theta_L - \tau$, whence

$$\tan\alpha\tan\beta - 2q\tan\beta + \ldots = \frac{\tan\theta_L - \tau - \frac{1}{3}\tau^3 \ldots}{1+\tan\theta_L(\tau + \frac{1}{3}\tau^3 + \ldots)}$$

$$= \tan\alpha\tan\beta - \tau(1+\tan^2\alpha\tan^2\beta) + \tau^2(\ ).$$

Since the radius of $ODEL$ is $\tan\beta$, we have

$$\mathrm{arc}ML = 2\tau\tan\beta = \frac{4q\tan^2\beta}{1+\tan^2\alpha\tan^2\beta} + \ldots.$$

Determining the angle $LON \equiv \theta_N - \theta_L$, we find similarly that

$$\mathrm{arc}LN = 2\cot\alpha(LON) = \frac{4q\tan^2\beta}{1+\tan^2\alpha\tan^2\beta} + \ldots.$$

Hence

$$\lim_{q \doteq 0} \frac{\mathrm{arc}ML}{\mathrm{arc}LN} = 1,$$

so that the families of circles (1) and (2) form an isothermal system. Also

$$\mathrm{arc}LN = q\rho_L{}^2 + \ldots.$$

$$\lim_{q \doteq 0} \frac{\mathrm{arc}LN}{\mathrm{arc}PQ} = \frac{2}{\tan^2\alpha + \cot^2\beta} = \tfrac{1}{2}\rho_L{}^2.$$

A curviliner quadrilateral, whose angles are all $\pi/2$ and the limits of the ratios of whose sides are all unity as q approaches zero, may be designated as an "infinitesimal square." We may state our result in the following terms:

The two families of circles (1) *and* (2) *form a network of infinitesimal squares whose sides are proportional to the squares of the distances from the origin to their nearest corner points.*

4. All systems of isothermal curves may be obtained by function-theory as follows: Set

$$X + iY = \theta(x+iy), \quad X - iY = \bar{\theta}(x-iy).$$

$$\therefore X = \tfrac{1}{2}[\theta(x+iy) + \bar{\theta}(x-iy)] \equiv U(x, y), \qquad Y \equiv \frac{-i}{2}(\theta - \bar{\theta}) \equiv V(x, y).$$

The lines $X=a$, $Y=b$ form an isothermal system. Under the transformation $X=U(x, y)$, $Y=V(x, y)$, the straight lines correspond to the curves $U(x, y)=a$, $V(x, y)=b$. The point (a, b) corresponds to the intersection P_1 of

$U=a$, $V=b$; the point $(a, b+q)$ corresponds to the intersection P_2 of $U=a$, $V=b+q$; the point $(a+q, b)$ corresponds to the intersection P_3 of $U=a+q$, $V=b$. It may be proved that angle $P_3P_1P_2=\frac{1}{2}\pi$, and that

$$\underset{q\doteq 0}{\text{limit}}\ \frac{P_1P_2}{P_1P_3}=1.$$

To obtain the correspondence in which the sense of the angles is reversed, we set $X-iY=\theta(x+iy)$.

5. Isothermal systems may be treated from the standpoint of transformation-groups.* For the case of the orthogonal circles (§3), the system may be derived from the system of lines parallel to the axes by the familiar transformation through reciprocal radii vectors.† The latter transforms any isothermal system into an isothermal system since it leaves invariant the partial differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0,$$

satisfied by the function $U\equiv U(x, y)$ of §4.

The University of Chicago, February, 1901.

ATMOSPHERIC REFRACTION.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The object of this article is not to set forth a new theory but simply to review the old. Let x=radius vector from center of the earth to any point in the path of the ray, φ=the angle between the ray and its normal, μ=the index of refraction, a, θ, μ_0=the values of x, φ, μ at the earth's surface, r=the atmospheric refraction.

Then $\mu x\sin\varphi=\mu_0 a\sin\theta$....(1).

Let φ'=the angle a consecutive element of the ray's path makes with the normal. Then $\varphi-\varphi'=dr$. By the laws of refraction

$$\mu\sin\varphi=(\mu+d\mu)\sin(\varphi-dr)=(\mu+d\mu)(\sin\varphi-dr\cos\varphi)$$

since $\sin dr=dr$ and $\cos dr=1$.

$\therefore\ d\mu\sin\varphi=\mu dr\cos\varphi$.

$\therefore\ dr=\dfrac{1}{\mu}\tan\varphi d\mu$, and $r=\displaystyle\int\frac{\tan\varphi d\mu}{\mu}$.

*Lie-Scheffers, *Differentialgleichungen*, pp. 156-157.

†Lie-Scheffers, *Geometrie der Beruhrungstransformationen*, pp. 6-9.

According to Simpson the law of decrease of density of the atmosphere is such that some power of the refractive index varies inversely as the distance from the center of the earth.

$$\therefore \left(\frac{\mu}{\mu_0}\right)^{n+1}=\frac{a}{x}.$$

This in (1) gives $\sin\varphi=\left(\frac{\mu}{\mu_0}\right)^n\sin\theta$.

$$\therefore \tan\varphi=\frac{\mu^n\sin\theta}{\sqrt{(\mu_0{}^{2n}-\mu^{2n}\sin^2\theta)}}.$$

$$\therefore r=\int_1^{\mu_0}\frac{\mu^{n-1}\sin\theta d\mu}{\sqrt{(\mu_0{}^{2n}-\mu^{2n}\sin^2\theta)}}=\frac{1}{n}\left[\theta-\sin^{-1}\left(\frac{\sin\theta}{\mu_0{}^n}\right)\right].$$

$\therefore \sin\theta=\mu_0{}^n\sin(\theta-nr)\ldots.(2)$.

Since nr is small, we write $\cos nr=1$, $\sin nr=nr$.

$\therefore \sin\theta=\mu_0{}^n(\sin\theta-nr\cos\theta)$.

$$\therefore r=\frac{(\mu_0{}^n-1)\tan\theta}{n\mu_0{}^n}\ldots.(3).$$

$$\text{From (2) }\frac{\sin\theta}{\sin(\theta-nr)}=\mu_0{}^n.$$

$$\therefore \frac{\sin\theta-\sin(\theta-nr)}{\sin\theta+\sin(\theta-nr)}=\frac{\mu_0{}^n-1}{\mu_0{}^n+1}.$$

$$\therefore \tan\frac{nr}{2}=\frac{\mu_0{}^n-1}{\mu_0{}^n+1}\tan\left(\theta-\frac{nr}{2}\right)\ldots.(4).$$

It has been demonstrated by the experiments of Biot and Arago that $\mu_0^2-1=4k\rho$, where ρ is the density of the atmosphere at the earth's surface and k is a constant so small that k^2 can be neglected.

Hence $\mu_0{}^n=(1+4k\rho)^{\frac{1}{2}n}=1+2nk\rho\ldots.(5)$.

This in (4) gives

$$\tan\frac{nr}{2}=\frac{nk\rho}{1+nk\rho}\tan\left(\theta-\frac{nr}{2}\right).\quad \text{Now }\tan\frac{nr}{2}=\frac{nr}{2}.$$

$$\therefore r=\frac{2k\rho}{1+nk\rho}\left(\frac{\tan\theta-\frac{1}{2}(nr)}{1+\frac{1}{2}(nr)\tan\theta}\right).$$

$$\therefore r^2+\frac{2r(1+nk\rho)}{n\tan\theta}=\frac{4k\rho}{n}.$$

$$r=\frac{\sqrt{(4nk\rho\tan^2\theta+1+2nk\rho)}-(1+nk\rho)}{n\tan\theta}\dots.(6).$$

(5) in (3) gives $r=\dfrac{2k\rho\tan\theta}{1+2nk\rho}\dots.(7).$

We must divide (6) and (7) by $\sin 1''$ to reduce to seconds.

$$r=\frac{\sqrt{(4nk\rho\tan^2\theta+1+2nk\rho)}-(1+nk\rho)}{n\tan\theta\sin 1''}\dots.(8).$$

$$r=\frac{2k\rho\tan\theta}{(1+2nk\rho)\sin 1''}\dots.(9).$$

When $\theta=\frac{1}{2}\pi$, (8) becomes $r=\sqrt{\dfrac{4k\rho}{n}}.\dfrac{1}{\sin 1''}\dots.(10).$

θ is the zenith distance. For 0^0 *C.* and 760 *MM.* experiment shows $4k=$ .000588768, and $\rho=1$.

From (10), $n=\dfrac{4\kappa\rho}{(r\sin 1'')^2}$, but $r=34'\ 30''=2070'$ when $\theta=\frac{1}{2}\pi$. This is the mean value for many observations.

$$\therefore\ n=\frac{.000588768}{(2070\times.000004848)^2}=5.8463.$$

n and $k\rho$ in (9) and (8) give $r=60.6\tan\theta\dots.(11).$

$$r=\frac{\sqrt{(1.00172106+.00344211\tan^2\theta)}-1.00086053}{.00002834\tan\theta}.$$

$$\therefore\ r=\frac{35316.17[\sqrt{(1+.003436\tan^2\theta)}-1]}{\tan\theta}\dots.(12).$$

$$=60.67\tan\theta-.0522\tan^3\theta\dots.(13).$$

(12) is applicable for all values of θ. (11) and (13) are accurate enough for values of θ up to 70° only.

Let g=gravity at surface, p=pressure, and δ=density of air at a distance x from center of earth. n is, also, found as follows:

$$dp=-\frac{a^2g\delta dx}{x^2}=ag\delta d\frac{a}{x}.$$

$$\text{Now }\frac{a}{x}=\left(\frac{\mu}{\mu_0}\right)^{n+1}=\left(\frac{1+4k\delta}{1+4k\rho}\right)^{\frac{1}{2}(n+1)}=1-2(n+1)k(\rho-\delta).$$

$\therefore\ d(a/x)=2k(n+1)d\delta.$

$\therefore\ dp=2k(n+1)ag\delta d\delta,\ p=agk(n+1)\delta^2.$

Let l=height of a homogenous atmosphere of density ρ exerting a pressure p_0.

$\therefore\ p_0=g\rho l.$

$$\therefore \frac{p}{p_0}=(n+1)\frac{a}{l}.\frac{k\delta^2}{\rho}, \text{ when } p=p_0,\ \delta=\rho.$$

$$\therefore\ 1=(n+1)\frac{a}{l}.k\rho, \text{ or } n=\frac{l/a}{k\rho}-1.$$

a=6366738 meters, $k\rho$=.000147192.

l=.76 (as many times as mercury is as heavy as air)=7993.15 meters.

$\therefore\ l/a$=.00125545.

$$\therefore\ n=\frac{.00125545}{.000147192}-1=7.5, \text{ a value of } n \text{ too large.}$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If the driving-wheels of Locomotive No. 200 on the Pennsylvania Railroad, m=7 feet in diameter, turn n=20 times in p=3 seconds, and lose r=12% of their forward motion by slipping on the smooth steel rails, at what rate per hour is the locomotive moving over the rails?

Solution by W. P. WEBBER, Mississippi Normal College, Houston, Miss.

$m\pi$=circumference of wheel, $mn\pi$=number of times the circumference is applied to the rail, and is therefore the distance the engine travels without slipping in p seconds.

Hence, $\dfrac{mn\pi}{p}$=distance engine travels in 1 second without slipping, and $\dfrac{mn(100-r)\pi}{100p}$=actual distance engine travels in 1 second, since it slips back r% of the distance it travels.

Substituting numbers for the letters, we have for the distance the engine travels in 1 second, $\dfrac{7\times20(100-12)\pi}{100\times3}$ feet.

Hence, the rate of the engine in miles per hour is

$$\frac{7\times20\times88\times\pi\times3600}{100\times3\times5280}=28\pi \text{ miles.}$$

ALGEBRA.

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123. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}}=2\sqrt[4]{\frac{1-a^2}{(1+a)^2}}, \text{ and } \sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}.$$

[*Haddon.*]

Solution by JOHN A. VAN GROOS, Fellow of Mathematics, University of Oregon, Eugene; Ore.; R.L.MOORE, Student in University of Texas, 2206 San Marcos Street, Austin, Tex,; and G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1). Multiply $\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}}=2\sqrt[4]{\frac{1-a^2}{(1+a)^2}}=2\sqrt[4]{\frac{1-a}{1+a}}$ through by

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}. \quad \therefore \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}-2\left(\frac{1-a}{1+a}\right)^{\frac{1}{4}}\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\left(\frac{1-a}{1+a}\right)^{\frac{1}{2}}=0.$$

$$\therefore \left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}-\left(\frac{1-a}{1+a}\right)\right]^2=0. \quad \therefore \frac{1+x}{1-x}=\frac{1-a}{1+a}.$$

$\therefore x=-a.$

(2). $\sqrt{(a^2-x^2)}=a^2\sqrt{(1-x^2)}-x\sqrt{(a^2-1)}.$

$$a^2-x^2=a^4-a^4x^2-2a^2x\sqrt{[(1-x^2)(a^2-1)]}+a^2x^2-x^2.$$

$\therefore 2a^2x\sqrt{[(1-x^2)(a^2-1)]}=a^2(1-x^2)(a^2-1).$

$\therefore \{2x-\sqrt{[(1-x^2)(a^2-1)]}\}\sqrt{[(1-x^2)(a^2-1)]}=0.$

$\therefore 4x^2=(1-x^2)(a^2-1)$ or $x^2=1.$

$\therefore x=-1,\ +1,$ or $\pm\sqrt{\frac{a^2-1}{3+a^2}}.$

The values $x=-1$ and $\sqrt{\frac{a^2-1}{3+a^2}}$ are the values satisfying the equation as given.

124. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A certain quantity of alcohol diluted with water so that in one liter there are c liters of pure alcohol, is mixed n times successively with p times the quantity of alcohol diluted so that 1 liter contains a liter of pure alcohol. How much pure alcohol does one liter of the nth mixture contain ?

Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Total put in.	*Alcohol put in.*	*Total.*
1	c	1
p	ap	$1+p$
$(1+p)p$	$(1+p)ap$	$(1+p)^2$
$(1+p)^2p$	$(1+p)^2ap$	$(1+p)^3$
$\vdots$	$\vdots$	$\vdots$
$(1+p)^np$	$(1+p)^nap$	$(1+p)^{n+1}$

The sum of the quantities in the second column is $c-a+a(1+p)^{n+1}$, and this divided by $(1+p)^{n+1}$ is the answer.

GEOMETRY.

151. Proposed by FRANK A. GIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

A point at a distance of 1 inch, 2 inches, and $2\frac{1}{2}$ inches, respectively, from three corners of a square. Construct the square. Also solve for the general distances a, b, c.

II. Solution by MARCUS BAKER, U. S. Coast Survey, Washington, D. C.

In the June-July number of the MONTHLY was published an *analysis* of this *without construction* as called for. The following is presented as an analysis *and* construction.

This is a variation of Rutherford's problem, which is: Given the distances a, b, c, of any point in the plane of an equilateral triangle, from the vertices of that triangle required to construct it. In the problem here proposed a right angled isosceles triangle replaces the equilateral triangle of Rutherford's problem.

Analysis. The triangle ABC is half of a square whose side is x, *i. e.* it is an isosceles triangle right angled at A. Let P be any point. Join P to A, B, and C by the lines a, b, and c, dividing the original triangle into three triangles. Now imagine each of these three triangles folded over its corresponding side of the original triangle in such wise that P falls at P_a, P_b, P_c. Complete the figure. Then there results a pentagon whose area is obviously twice that of ABC, *i. e.* equals x^2. This pentagon is composed of three triangles, viz:

P_cP_aB,	sides b, b, $b\sqrt{2}$,	area$=\frac{1}{2}b^2$
P_aP_bC,	sides c, c, $c\sqrt{2}$,	area$=\frac{1}{2}c^2$
$P_aP_bP_c$,	sides $b\sqrt{2}$, $c\sqrt{2}$, $2a$,	area$=$

$$\left[\left(\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}+a\right)\left(-\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}+a\right)\left(\frac{b}{\sqrt{2}}-\frac{c}{\sqrt{2}}+a\right)\left(\frac{b}{\sqrt{2}}+\frac{c}{\sqrt{2}}-a\right)\right]^{\frac{1}{2}}$$

Construction. From the above analysis it is obvious that we must first

construct the triangle $P_aP_bP_c$, whose sides are known, and then upon $b\sqrt{2}$ and $c\sqrt{2}$ construct isosceles right triangles. The vertices of these triangles (at the right angles) and the middle of the side $2c$ are the vertices of the required triangle.

Number of Solutions. In constructing $P_aP_bP_c$ we may take for its sides

	$a\sqrt{2}$,	$b\sqrt{2}$,	$2c$ ....(1).
or	$a\sqrt{2}$,	$2b$,	$c\sqrt{2}$ ...(2).
or	$2a$	$b\sqrt{2}$,	$c\sqrt{2}$....(3).

In the special problem before us $a=1$, $b=2$, $c=2\frac{1}{2}$; whence

1.414,	2.828,	5. ....(1).
1.414,	4. ,	3.535....(2).
2, ,	2.828,	3,535....(3).

In the first case there is no real solution.

In the second case P falls *without* the triangle, and $x=2\frac{7}{16}$.

In the third case P falls *within* the triangle, and $x=2\frac{13}{16}$.

The values are derived by scaling off from the figures. The construction here given is for the third case, the unit being one centimeter.

154. Proposed by **ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.**

The angle between the edge of a trihedral angle and the bisector of the opposite face angle is less than, equal to, or greater than half the sum of the other two face angles, according as it is itself acute, right, or obtuse.

Solution by the **PROPOSER.**

Let $S-ABC$ be a trihedral angle and SD the bisector of the face angle ASB.

CASE I. $\angle DSC<$a right angle. (See Fig. 1.)

To prove $\angle DSC<\frac{1}{2}(\angle ASC+\angle BSC)$.

From C, any point of the edge SC, draw CD perpendicular to SD.

Through D, in the face ASB, draw AB perpendicular to SD. Then, SD is perpendicular to the plane ABC.

Connect S with F and E.

Comparing the right triangles AFD and BED, $AD=BD$ (since triangle ASD=triangle BSD), and the vertical angles at D are equal. Hence, the triangles are equal, and $DE=DF$.

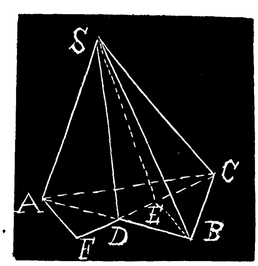


Fig. 1.

It follows that right triangles FSD and ESD are equal, and $\angle FSD=\angle ESD$.

Now, since SD is perpendicular to the plane ABC, the plane DSC is perpendicular to the plane ABC. Therefore BE and AF, which lie in one of these planes and are perpendicular to their intersection, are perpendicular to the other plane, DSC.

Consequently, SE and SF are the projections of SB and SA, respectively, on the plane DSC.

Therefore $\angle ESC<\angle BSC$, and $\angle FSC<\angle ASC$; or $\angle DSC-\angle DSE<\angle BSC$, and $\angle DSC+\angle DSF<\angle ASC$.

Adding, and remembering that $\angle DSE=\angle DSF$, $2\angle DSC<\angle BSC+\angle ASC$, $\angle DSC<\frac{1}{2}(\angle BSC+\angle ASC$.

CASE II. $\angle DSC=$a right angle. (See Fig. 2.)

To prove $\angle DSC=\frac{1}{2}(\angle ASC+\angle BSC)$.

Draw DX parallel to SC, and AB perpendicular to SD.

Then SD is perpendicular to DX, and, hence, to the plane determined by AB and DX.

This plane intersects the planes of faces BSC and ASC in BE and AF, respectively.

Since SC is parallel to DX, it is parallel to the plane BDX, and, hence, parallel to BE and AF.

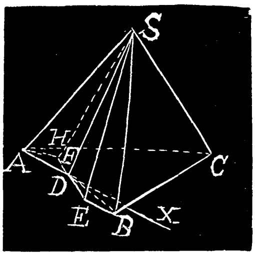


Fig. 2.

Through SD pass a plane perpendicular to SC, intersecting the plane of AB and DX in EF, and the planes of faces BSC and ASC in SE and SF, respectively. Since CS is perpendicular to the plane FSE, so are BE and AF.

Hence, $\angle$'s BES, BED, AFS, and AFD are right angles.

Right triangles DAF and DBE are equal, since $AD=BD$ (from equality of right triangles ASD and BSD), and the vertical angles at D are equal.

Therefore $BE=AF$. Hence, since $SB=SA$, right triangles SEB and SFA are equal, and $\angle ASF(=\angle ASH)=\angle BSE$.

Now, since $\angle$'s CSD, CSE, and CSH are right angles, $\angle CSD=\frac{1}{2}(\angle CSE+\angle SCH)$, $=\frac{1}{2}(\angle CSB+\angle BSE+\angle CSA-\angle ASH)$
$=\frac{1}{2}(\angle CSB+\angle CSA)$.

CASE III. $\angle DSC>$a right angle. (See Fig. 3.)

To prove $\angle DSC>\frac{1}{2}(\angle CSA+\angle CSB)$.

Produce AS, BS, and DS, forming another trihedral angle $S-A'B'C$.

By Case I, $\angle CSD'<\frac{1}{2}(CSA'(\angle CSA'+\angle CSB')$, or $180^\circ-\angle CSD<\frac{1}{2}(180^\circ-\angle CSA+180^\circ-\angle CSB)$, from which $-\angle CSD<-\frac{1}{2}(\angle CSA+\angle CSB)$.

Therefore, $\angle CSD>\frac{1}{2}(\angle CSA+\angle CSB)$.

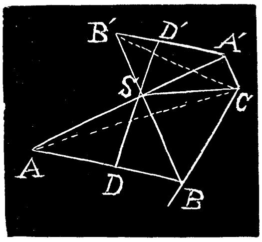


Fig. 3.

Also solved by *H. C. WHITAKER*.

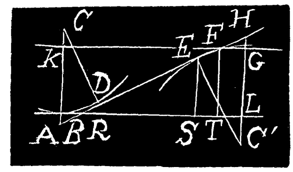
155. Proposed by J. C. NAGLE, M.A., C.E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Texas.

A special case of the following problem was sent me some time ago by an ex-member of one of my engineering classes, as occurring on the Southern Pacific Ry., near Devil's River:

Two straight tracks, p feet between centers, are to be united by a cross-over composed of two curves of radius R, and a length L of intervening tangent. Required the central angles and the distance between tangent points, measured along main track. In the special case referred to p was 62 feet, L 100 feet with 9° 30 curves.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\tan OCD = \tan EC'G = m$, $OC = CD = C'G = C'E = R$, $DE = L$, $KO = LG = p$. The equation to the line AH is $y = mx - b$, where O is origin and $b = OA$.



$CA = CD\sec OCD$. $\therefore R + b = R\sqrt{(1+m^2)}$.

$\therefore y = mx + R - R\sqrt{(1+m^2)}$.

When $y=0$, $x = R[\sqrt{(1+m^2)} - 1]/m$,

When $y=p$, $x = [p + R\sqrt{(1+m^2)} - R]/m = x_1$.

$\therefore OB = BD = EF = FG = R[\sqrt{(1+m^2)} - 1]/m$.

Since $BF = \sqrt{[BT^2 + TF^2)} = \sqrt{[(x_1 - x)^2 + p^2]}$,

$\therefore L + 2R[\sqrt{(1+m^2)} - 1]/m = \sqrt{[(p/m)^2 + p^2]} = (p/m)\sqrt{(1+m^2)}$.

$\therefore [(2R-p)^2 - L^2]m^2 + 4RLm = 4Rp - p^2$. This determines m.

$OL = OT + TL = x + x_1 = [p + 2R\sqrt{(1+m)^2} - 2R]/m$.

$BR = BD\cos OCD = x/\sqrt{(1+m^2)} = R[\sqrt{(1+m^2)} - 1]/[m\sqrt{(1+m^2)}]$.

$RS = BT - 2BR = p/m - 2R[\sqrt{(1+m^2)} - 1]/[m\sqrt{(1+m^2)}]$.

$p = 62$, $L = 100$, $R = 604$.

$\therefore 325829m^2 + 60400m = 36487$ or $m = .254559$.

$\therefore \angle OCD = 14° \; 16' \; 52.6'' =$ central angles.

$OL = 394.9$ feet, $RS = 96.9$ feet.

CALCULUS.

112. Proposed by J. SCHEFFER. A. M., Hagerstown, Md.

A sphere of radius r is pierced by a cylinder radius $\frac{1}{2}r$ so that the cylinder just grazes the center of the sphere. Find volume removed; the lateral surface and the spherical surface removed.

Solution by L. C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and the PROPOSER.

Taking the center of the sphere for coördinates, we have for the equations of sphere and cylinder, respectively,

$$x^2 + y^2 + z^2 = r^2, \; y^2 = rx - x^2.$$

Therefore volume removed, $V = 4\int_0^r dx \int_0^{\sqrt{(rx-x^2)}} z dy$

$$= 2\int_0^r \left[(r^2 - x^2)\sin^{-1}\sqrt{\frac{x}{r+x}}dx + \sqrt{(rx)}(r-x)dx \right] = \frac{3\pi - 4}{9}.2r^3.$$

The lateral surface $S = 2\int z\partial s$, s being an arc of the base circle of the cylinder. From $y^2 = rx - x^2$,

$$\frac{dy}{dx}=\frac{r-2x}{2\sqrt{(rx-x^2)}},\quad \frac{dz^2}{dx^2}=1+\left(\frac{dy}{dx}\right)^2=\frac{r^2}{4(rx-x^2)},\quad z^2=r^2-x^2-y^2=r^2-rx.$$

$$\therefore\ L=2r\sqrt{r}\int_0^r\frac{dx}{\sqrt{x}}=4r^2.$$

The spherical surface removed, $S=4\int_0^r dx\int_0^{\sqrt{(rx-x^2)}}dy\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}$

$$=4r\int_0^r dx\int_0^{\sqrt{(rx-x^2)}}\frac{dy}{\sqrt{(r^2-x^2-y^2)}}=4r\int_0^r \sin^{-1}\sqrt{\frac{x}{r+x}}.dx=(\pi-2)2r^2.$$

NOTE. The results of Problem 6 in Byerly's *Integral Calculus*, page 282, 2nd Ed. (1898) are correct, if b represents the diameter of the cylindrical hole instead of the radius.

Also solved by *H. C. WHITAKER.*

113. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

At what rate per unit of time are the roots of the equation $(x^2+px+q=0)$ changing, if $p=mq$ and q varies uniformly at the rate of 1/12 per unit of time, when $p=12$ and m remains constant?

Solution by H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

The roots of the given equation are

$$x=\tfrac{1}{2}[-mq\pm\sqrt{(m^2q^2-4q)}],$$

$$dx=\tfrac{1}{2}[-m\pm(m^2q-2)(m^2q^2-4q)^{-\frac{1}{2}}]dq.$$

When $q=12\div m$ and $dq=\frac{1}{12}$,

$$dx=\tfrac{1}{24}\left[-m\pm\frac{(6m-1)\sqrt{m}}{2\sqrt{(9m-3)}}\right].$$

Also solved by *WILLIAM HOOVER.*

114. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

If two concentric ellipses have equal axes inclined at an angle ω, their common area is

$$A=2ab\tan^{-1}\left(\frac{2ab}{(a^2-b^2)\sin\omega}\right).$$

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and J. B. GREGG, Senecaville, Ohio.

Let AB be the semi-major axis of one ellipse, and AB' that of the other, MAC and NAD being common diameters. F is the intersection of AB with the second ellipse, and E and G are the intersections of major axis of the second ellipse with the first.

Let angle $B'AB=\omega$. It can be easily shown that angle $CAF=\frac{1}{2}\omega$, and angle $DAF=$angle $DAG=\frac{1}{2}\pi-\frac{1}{2}\omega$.

The polar equation of an ellipse referred to the center as pole is

$$r^2=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

Since section $EAC=CAF$, and $DAF=DA'G$; we have for $ECFA$ integral

$$a^2b^2\int_{\frac{1}{2}\omega}^{\omega}\frac{d\theta}{a^2\sin^2\theta+b^2\cos^2\theta}=ab\left[\tan^{-1}\left(\frac{a}{b}\tan\omega\right)-\tan^{-1}\left(\frac{a}{b}\tan\tfrac{1}{2}\omega\right)\right]$$

$$=ab\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega+b^2\cos\omega}\right).$$

To find the area of $AFDG$ we have only to put $\pi-\omega$ in place of ω, and thus we get

$$ab\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega-b^2\cos\omega}\right).$$

$\therefore$ Area of $NCDM$

$$=2ab\left[\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega+b^2\cos\omega}\right)+\tan^{-1}\left(\frac{ab\sin\omega}{a^2\sin^2\omega+b^2\cos^2\omega-b^2\cos\omega}\right)\right]$$

$$=2ab\tan^{-1}\left[\frac{2ab}{(a^2-b^2)\sin\omega}\right].$$

Also solved by *G. B. M. ZERR.*

115. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The axes of two right elliptic cylinders intersect at right angles in such a manner that the *major axes of the sections* are perpendicular. Supposing the axes to be $(A, B)>(a, b)$, what is the common volume ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x^2/B^2+z^2/A^2=1$, $x^2/a^2+y^2/b^2=1$, be the equations to the cylinders.

$$\therefore\ V=\frac{8Ab}{aB}\int_0^B\sqrt{[(a^2-x^2)(B^2-x^2)]}dx,\ B<a\ldots\ (1).$$

$$V=\frac{8Ab}{aB}\int_0^a\sqrt{[(B^2-x^2)(a^2-x^2)]}dx,\ a<B\ldots.(2).$$

Let $x=B\sin\theta$ in (1).

Then $V=8ABb\int_0^{\frac{1}{2}\pi}\sqrt{[1-(B^2/a^2)\sin^2\theta]}\cos^2\theta d\theta$

$$=\frac{8Aa^2b}{3B}\{[1+B^2/a^2]E[B/a,\ \tfrac{1}{2}\pi]-[1-B^2/a^2]F[B/a,\ \tfrac{1}{2}\pi]\}.$$

Let $x=a\sin\theta$ in (2).

Then $V=8Aab\int_0^{\frac{1}{2}\pi}\sqrt{[1-(a^2/B^2)\sin^2\theta)}\cos^2\theta d\theta$

$$=\frac{8AB^2b}{3a}\{[1+a^2/B^2]E[a/B,\ \tfrac{1}{2}\pi]-[1-a^2/B^2]F[a/B,\ \tfrac{1}{2}\pi]\}.$$

MECHANICS.

121. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. Gloucestershire, England.

Two equal scale pans of mass m hang at rest over a smooth pulley. An inelastic particle, mass M, is dropped from a height h into one pan, and simultaneously another of equal mass and elasticity e is dropped from the same heignt into the other. Prove that every impact occurs when the pans are in their original positions, and find the total space described by either pan before motion ceases.

Soluiion by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The velocity of each particle just before impact$=\sqrt{(2gh)}$.

The velocity of first rebound of elastic particle$=e\sqrt{(2gh)}$.

The velocity of second rebound of elastic particle$=e^2\sqrt{(2gh)}$.

The velocity of nth rebound of elastic particle$=e^n\sqrt{(2gh)}$.

The elastic particle imparts a velocity to the scale pans at first impact $=\frac{M(1+e)}{2m+M}\sqrt{(2gh)}$, at the second impact a velocity$=\frac{Me(1+e)}{2m+M}\sqrt{(2gh)}$, at the nth impact a velocity$=\frac{Me^{n-1}(1+e)}{2m+M}\sqrt{(2gh)}$.

The inelastic particle imparts a velocity to the scale pans at first impact $=\frac{M}{2m+M}\sqrt{(2gh)}$.

Resultant velocity of these two particles at first impact$=\frac{Me}{2m+M}\sqrt{(2gh)}$.

The acceleration caused by inelastic particle$=\frac{Mg}{2m+M}$.

The time required for the scale pans to return to their original positions

$$=\frac{2Me}{2m+M}\sqrt{(2gh)}/\frac{Mg}{2m+M}=2e\sqrt{(2h/g)}.$$

The time required for the elastic particle to return to the same position $2e\sqrt{(2gh)}/g=2e\sqrt{(2h/g)}$.

Therefore, second impact takes place in original position.

As the scale pans have a velocity $=\frac{Me}{2m+M}\sqrt{(2gh)}$, the resultant velocity after the second impact

$$=\frac{Me(1+e)}{2m+M}\sqrt{(2gh)}-\frac{Me}{2m+M}\sqrt{(2gh)}=\frac{Me^2}{2m+M}\sqrt{(2gh)}.$$

The time required for the scale pans to return to their original position

$$=\frac{2Me^2}{2m+M}\sqrt{(2gh)}/\frac{Mg}{2m+M}=2e^2\sqrt{(2h/g)}.$$

It takes the elastic particle a time $=2e^2\sqrt{(2gh)}/g=2e^2\sqrt{(2h/g)}$.

Therefore, the third impact takes place in original position.

Proceeding thus, we find the resultant velocity after the nth impact $\frac{Me^n}{2m+M}\sqrt{(2gh)}$, and the time required for the scale pans to return to the original position $=2e^n\sqrt{(2h/g)}$.

Therefore, every impact occurs in the original position.

Space passed over by scale pans between nth and $(n+1)$th impact

$$\frac{Mg}{2m+M}\times\frac{2he^{2n}}{g}=\frac{2Mhe^{2n}}{2m+M}.$$

Therefore, total space passed over $=\frac{2Mh}{2m+M}(e^2+e^4+e^6+e^8+\ldots.)$

$\therefore\ S=\frac{2Mh}{2m+M}\cdot\frac{e^2}{1-e^2}$.

The elastic particle passes over a space S where

$$S=h+2he^2+2he^4+2he^6+\ldots.=\frac{1+e^2}{1-e^2}h.$$

The time required for scale pans to come to rest

$$=2\sqrt{\left(\frac{2h}{g}\right)}(e+e^2+e^3+e^4+\ldots.)=2\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{e}{1-e}.$$

The time required for the elastic particle to come to rest

$$=2\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{e}{1-e}+\sqrt{\left(\frac{2h}{g}\right)}=\sqrt{\left(\frac{2h}{g}\right)}\cdot\frac{1+e}{1-e}.$$

DIOPHANTINE ANALYSIS.

85. Proposed by A. H. BELL, Hillsboro, Ill.

Given $x^2-85\frac{1}{4}y^2=5$. What is the value of x and y in whole numbers ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$x^2-85\frac{1}{4}y^2=5$, $4x^2-341y^2=20$, $x^2=5+\frac{341}{4}y^2$.

Let $y=2z$. $\therefore x^2=5+341z^2$.

Let $z=2$. Then $x=37$, $y=4$ are the least integral values.

86. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that $x^2+1457\equiv0$ (mod. 2389) is insoluble.

Solution by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y., and H. S. VANDIVER, Bala, Pa.

We are to prove that -1457 is a quadratic non-residue of the odd prime 2389. Using Legendre's symbol, we must show that

$$\left(\frac{-1457}{2389}\right)=-1.$$

We have $\left(\frac{-1457}{2387}\right)=\left(\frac{-1}{2389}\right)\left(\frac{31}{2389}\right)\left(\frac{47}{2389}\right)$.

$\left(\frac{-1}{2389}\right)=+1$, since 2389 is of the form $4n+1$.

By the law of reciprocity, we have $(\frac{31}{2389})=(\frac{2389}{31})=(\frac{2}{31})=+1$, since in the first place 2389 is of the form $4n+1$, and since, secondly, 31 is of the form $8n\pm1$.

$$\left(\frac{47}{2389}\right)=\left(\frac{2389}{47}\right)=\left(\frac{39}{47}\right)=-\left(\frac{47}{39}\right)=-\left(\frac{8}{39}\right)=-\left(\frac{2^2}{39}\right)\left(\frac{2}{39}\right)=-\left(\frac{2}{39}\right)=-1,$$

since 47 and 39 are both of the form $4n-1$, and 31 is of the form $2n\pm1$.

Therefore, finally, $\left(\frac{-1457}{2389}\right)=(+1)(+1)(-1)=-1$.

Also solved by *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

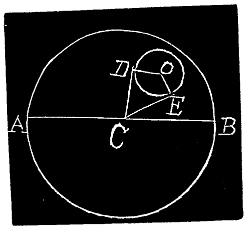
103. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

Let C be the center of the semi-circle, O the center of the random circle, CD, CE tangents from C to random circle.

Also let $AC=r$, $CO=x$, $OD=z=OE$, $\angle ACO=\theta$, $\angle DCO=\varphi$, p=required chance.



The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $r/(1+\sin\theta)=x'$, and $r/(1+\sin\theta)=x'$ and r; of z, 0 and $x\sin\theta$, and 0 $r-x$.

Then $p=2\varphi/\pi$, $\varphi=\sin^{-1}(z/x)$.

$$\therefore\ p=\frac{\dfrac{2}{\pi}\displaystyle\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}\sin^{-1}(z/x)xdxdz+\int_0^{x'}\int_0^{x\sin\theta}\sin^{-1}(z/x)xdxdz\right]d\theta}{\displaystyle\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}xdxdz+\int_0^{x'}\int_0^{x\sin\theta}xdxdz\right]d\theta}$$

$$=\frac{72}{\pi r^3(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}\int_0^{r-x}\sin^{-1}(z/x)xdxdz+\int_0^{x'}\int_0^{x\sin\theta}\sin^{-1}(z/x)xdxdz\right]d\theta$$

$$=\frac{72}{\pi r^3(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[\int_{x'}^{r}[(r-x)\sin^{-1}(\frac{r-x}{x})+\sqrt{(2rx-r^2)}-x]xdx\right.$$

$$\left.+\int_0^{x'}(\theta\sin\theta+\cos\theta-1)x^2dx\right]d\theta$$

$$=\frac{4}{\pi(3\pi-4)}\int_0^{\frac{1}{2}\pi}\left[2-\frac{3\theta}{(1+\sin\theta)^2}-\frac{(2+3\sin\theta+\sin^2\theta)\cos\theta}{(1+\sin\theta)^3}\right]d\theta$$

$$=\frac{4}{\pi(3\pi-4)}(\pi-\tfrac{1}{4}-2\log2)=\frac{1}{3\pi-4}(4-\frac{1}{\pi}-\frac{8}{\pi}\log2).$$

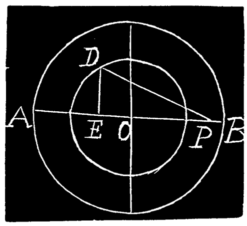
104. Proposed by **LON C. WALKER, A. M.**, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

In a given sphere two radii are drawn at random, and a point taken in each at random. (1) Find the chance that the distance between the two points does not exceed the radius of the sphere. (2) Find the distance between them.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be one of the points on radius OB, D the second point on radius OD. Let $OB=r$, $OP=x$, $OC=OD=y$. $OE=z$.

Then $PD=\sqrt{(x^2+y^2+2xz)}$.



The first point lies on the surface of a sphere radius x. An element of surface of this sphere is $2\pi xdx$. The second point lies on the surface of the sphere radius y. An element of surface of this sphere is $2\pi ydy$.

Let $(r^2-x^2-y^2)/2x=z'$, p=chance, and Δ=the average distance. Then

$$(1).\ p=\frac{1}{(\frac{4}{3}\pi r^3)\pi r^2}\int_0^r\int_0^r\int_{-y}^{z'}2\pi x dx.2\pi y dy dz=\frac{3}{2r^5}\int_0^r\int_0^r y[r^2-(x-y)^2]dxdy$$

$$=\frac{1}{8r^5}\int_0^r(3r^4-6r^2x^2+8r^3x)dx=\tfrac{5}{8}.$$

$$(2).\ \Delta=\frac{1}{(\frac{4}{3}\pi r^3)\pi r^2}\int_0^r 2\pi x dx\Big[\int_0^x\int_{-y}^{y}2\pi y\sqrt{(x^2+y^2+2xz)}dydz$$

$$+\int_x^r\int_{-y}^{y}2\pi y\sqrt{(x^2+y^2+2xz)}dydz\Big].$$

$$\therefore\ \Delta=\frac{1}{r^5}\int_0^r dx\Big[\int_0^x y[(x+y)^3-(x-y)^3]dy+\int_x^r y[(y+x)^3-(y-x)^3]dy\Big]$$

$$=\frac{1}{10r^5}\int_0^r(15r^4x+10r^2x^3-x^4)dx=\tfrac{59}{60}r.$$

Also solved by the *PROPOSER*.

105. Proposed by L. C. WALKER, Assistant in Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the average distance of the center of an ellipsoid, axes $2a$, $2b$, and $2c$ from its surface.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Different interpretations will result in different solutions. The method generally employed by the best thinkers in this department of mathematics is the following:

$$d=\frac{\int rds}{\int ds},$$ where ds is an element of surface.

Let s be the eighth part of the surface of the ellipsoid.

Then $d=\dfrac{1}{s}\int rds.$

$$ds=\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}dxdy=\left[\frac{a^4b^4-b^4(a^2-c^2)x^2-a^4(b^2-c^2)y^2}{a^2b^2(a^2b^2-b^2x^2-a^2y^2)}\right]^{\frac{1}{2}}dxdy$$

$$r=\sqrt{(x^2+y^2+z^2)}=\frac{1}{ab}[a^2b^2c^2+b^2(a^2-c^2)x^2+a^2(b^2-c^2)y^2]^{\frac{1}{2}}.$$

$$\therefore\ p=\int_0^a\int_0^{(b/a)\sqrt{(a^2-x^2)}}[c^2+x^2+y^2-\frac{c^2}{b^4}(b^2-c^2)y^2-\frac{c^2}{a^4}(a^2-c^2)x^2$$

$$+\frac{c^2(x^2+y^2)(b^4x^2+a^4y^2)}{a^2b^2(a^2b^2-b^2x^2-a^2y^2)}]^{\frac{1}{2}}\,dxdy.$$

This expression does not seem to be easy to integrate.

MISCELLANEOUS.

96. Proposed by H. M. CASH, Lore City, Guernsey County, Ohio.

A stick of timber is 12 feet long, 8 inches deep, and 3 inches wide at one end, and 5 inches deep, and 12 inches wide at the other end. An what distance from either end should it be cut to divide it into two equal parts?

Solution by H. C. WHITAKER, A.M., Ph.D., Manual Training School, Philadelphia, Pa., and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

By the Prismoidal Formula, Volume$=\frac{1}{6}$(upper base+lower base+4 middle section)$\times$height.

Hence, the volume of the whole stick equals

$$\tfrac{1}{6}(8\times3+5\times12+4\times\tfrac{13}{2}\times\tfrac{15}{2})\times144=6696.$$

Denote the distance of the required section from the 8×3 end by x, the depth at that section by y, the width by z. Then $y=8-\frac{1}{48}x$, and $z=3+\frac{1}{16}x$; also

$$\tfrac{1}{6}\left[8\times3+yz+4\left(\frac{8+y}{2}\right)\left(\frac{3+z}{2}\right)\right]x=3348.$$

Whence $x^3-504x^2-55296x+7713792=0$.

$x=84.8856$ inches$=7$ feet, 0.8856 inches.

Also solved by *R. L. MOORE*.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A spherical soap-bubble is electrified in such a way that the excess of the internal over the external air pressure is 2π when the bubble is in equilibrium. How does the tension of the film vary with the electric density?

Solution by the PROPOSER.

Let p be the excess of internal over external pressure, σ=electrical density, t=tension, r=radius of bubble.

Then $p+2\pi\sigma^2=2t/r$, (see Minchin, Vol. II, page 487, third edition).

But $p=2\pi$.

$\therefore 2\pi r(1+\sigma^2)=2t$, or $\dfrac{t}{1+\sigma^2}=\pi r$.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

148. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

According to his contract a professor is to receive $1800. in cash plus board, etc., for his services during a scholastic year of nine months. This sum is payable in equal installments of $200. at the end of each scholastic month. The treasurer, however, paid the professor in ten equal installments of $180. The last two installments were paid Monday and Thursday of the last week in the scholastic year. Regarding money worth 6%, out of how much was the professor defrauded by the wiley treasurer?

149. **Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.**

A wine cask contains 256 gallons of wine; a certain quantity is drawn off and the cask is filled with water; the same quantity of the mixture is drawn off and the cask again filled with water and so on for four draughts, when there remain only 81 gallons of wine in the cask. How many gallons of wine are drawn at each of the draughts?—[*Colburn's Algebra.*]

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

ALGEBRA.

145. **Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.**

Factorize $2b^2c^2+2c^2a^2+2a^2b^2+2a^2d^2+2b^2d^2+2c^2d^2-a^4-b^4-c^4-d^4$.

146. **Proposed by B. F. YANNEY, Professor of Mathematics, Mount Union College, Alliance, Ohio.**

If the series 1, 3, 5, ... $2n-1$, ... be divided into successive groups of r terms each, the sum of the terms of the nth group will be $(2n-1)$ times the sum of the terms of the first group, or $(2n-1)r^2$.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

GEOMETRY.

175. **Proposed by W. P. WEBBER, Mississippi Normal College. Houston, Miss.**

A field is enclosed by a fence in circular form and a straight gate 20 feet wide. The fence is 100 feet in length. How much land in the field? [Solution by most elementary method possible.]

176. **Proposed by R. A. WELLS, Franklin College, New Athens. Ohio.**

If there be three straight lines which meet in a point, and the arbitrary constants of their equations, expressed in the slope form, be taken as the coördinates of three points, these three points will lie in a straight line.

⁂ Solutions of these problem should be sent to B. F. Finkel not later than Dec. 10.

CALCULUS.

116. Proposed by M. E. GRABER, A. B., Tutor in Mathematics, Heidelberg University, Tiffin, Ohio.

Find the curve the length of whose arc measured from a given point is a mean proportional between the ordinate and twice the abscissa.

117. Proposed by WM. FRED FLEMING, Chicago, Ill.

A tin watering-pot is constructed by joining the frustums of two right cones, so that their intersection is a mathematical one, their axes meeting at an angle of 45°. The bases of the smaller frustum are 2 inches and 4 inches in diameter, its altitude 8 inches. The bases of the larger frustum are 10 inches and 12 inches in diameter, its altitude 15 inches. In joining the two frustums the edges of the two larger bases are brought into coincidence. Water is poured into the vessel until it begins to run out of the spout. How many gallons (231 cubic inches) are required? How much water is in the spout and how much in the can? The vessel is tilted forward (in the plane of the axes of the two frustums) sufficiently to allow one-half of the water to run out. How much of the liquid is left in the spout and can, and what is the area of the surface of the water in spout and can? Through what angle has the vessel been tilted?

⁎⁎⁎ Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

MECHANICS.

126. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, Stroud, London, England.

AB is the horizontal base of a smooth cycloidal tube, vertex downward. A sphere is placed in the tube at A, and when it reaches the vertex another sphere of different mass is placed in the tube at B. When and where do they meet, and find their velocity immediately after collision, the spheres being partially elastic.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Develop the Fourier Series to represent the temperature of a circular wire of uniform cross-section, in which the temperatures of the four quadrants are in order $t, 2t, 3t, 4t$.

128. Proposed by M. E. GRABER, A. B., Heidelberg University, Tiffin, Ohio.

A particle is placed on the convex side of a smooth ellipse and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the center. Find the position of equilibrium.

129. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics. Drury College, Springfield, Mo.

Two spheres whose masses are M_1 and M_2 are a units apart, and attract each other with a force$=M_1M_2/a^2$. Find work done in carrying a unit mass from the center point between them a distance r in a direction θ with line of centers.

⁎⁎⁎ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

DIOPHANTINE ANALYSIS.

91 Proposed by L. C. WALKER, A.M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find the three least positive integral numbers whose sum, sum of their squares, and sum of their cubes shall each be rational squares.

92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

AVERAGE AND PROBABILITY.

115. Proposed by L.C.WALKER, A.M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Three points are at random within a given triangle. Find the chance that they will all lie on one side of some one line that can be drawn through the center of gravity of the triangle.

116. Proposed by the late ENOCH BEERY SEITZ,

The average area of the quadrilateral formed by joining four radom points on the surface of a circle, radius a, is $\frac{4a^2}{3\pi}$.

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

MISCELLANEOUS.

117. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

If $x\cos\alpha+y\cos\alpha=a\cos\theta+b\cos\varphi$, and $x\sin\alpha+b\sin\varphi=y\sin\alpha+a\sin\theta=\kappa$, find the maximum value of κ, and the values of x and y.

118. Proposed by O. W. ANTHONY, New York, N. Y.

If f is determined by the equation $f(\mu\nu)=f(\mu)f^{-1}(\nu)+f(\nu)f^{-1}(\mu)$, when f^{-1} is the inverse of f, show that $f[(2)^\mu]=\frac{k^{\mu+1}}{2^{\mu+1}}$, where k is the constant.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

BOOKS AND PERIODICALS.

Annals of Mathematics. Published under the auspices of Harvard University. Second series, Vol. 2, No. 4. Price, $2.00 per year in advance. Published in October, January, April, and July.

The July number of the current year contains the following articles: Concerning du Bois-Reymond's Two Relative Integrability Theorems, by Professor E. H. Moore; On a Theorem of Kinematics, by Dr. P. Saurel; The Collineations of Space whichTrans forms a Non-Degenerate Quadratic Surface into Itself, by Dr. R. G. Wood; Note on Multiply Perfect Numbers, by Dr. J. Westlund; The Isoperimetrical Problem on Any Surface, by Mr. J. K. Whitmore; On a Surface of the Sixth Order which is Touched by the Axes of all Screws Reciprocal to Three Given Screws, by Professor E. W. Hyde; Note Sur l' évalution d'une intégral define, Par le Professor D. Sintsof. B. F. F.

The American Journal of Mathematics. Published under the auspices of the Johns Hopkins University and edited by Frank Morely with the coöperation of other mathematicians. Price, $5.00 per year in advance.

The October number contains the following articles: Memoir on the Algebra of

Symbolic Logic, by A. N. Whitehead; Secular Perterbations of the Planets, by G. W. Hill; Representation of Linear Groups as Transitive Groups, by Leonard E. Dickson; A Class of Number Systems in Six Units, by G. P. Starkweather. B. F. F.

Éléments de Mathématiqes Supérieur, A l'Usage des Physiciens, Chimistes et Ingénieurs et des Aspirants a ces Titres, des Conducteurs de Travaux, et des Élèves des Facultés des Scienceset des Écoles Industrielles. Par B. Vogt, Ancien Élève de l'École Normale Supérieur Agrégé de Sciences Mathématiques, Docteur de Sciences, Professeur a l'Université de Nancy. Paris, France, Librairie Nony te C[ie].

This work is divided into seven parts. The first part treats on Algebra; the second part on Analytical Geometry; the third part on Derivatives and Differentials; the fourth part on Theory of Equations; the fifth part on Geometric Applications; sixth part, Integral Calculus; seventh part, Differential Equations. Pages 485-565 are devoted to notes on Algebra and Analytic Geometry, in which are treated Series, Elimination, Transformation of Coördinates, Curves on Surfaces, etc. Pages 569-603 are devoted to exercises in the various subjects of which the book treats. We are sure that this work will be of great value to the practical mathematician as well as to the student of general mathematics.

B. F. F.

Linear Groups, With an Exposition of the Galois Field Theory. By Leonard Eugene Dickson, Ph. D., Assistant Professor of Mathematics in the University of Chicago. Large 8vo, cloth, 312 pages. Leipzig, Germany: B. G. Teubner.

This volumn is intended as an introduction to the subject of Linear Groups in a Finite Field, a subject having immediate application in many problems of Geometry and Function Theory. The earlier chapters of the book are devoted to an elementary exposition of the theory of Galois Fields chiefly in their abstract form, while the second part of the book is devoted to an elementary exposition of the more important results concerning Linear Groups in a Galois Field.

Dr.Dickson is the first American mathematician who has written a book on the Theory of Groups, and in this work he has added a valuable contribution to the literature of the subject. B. F. F.

Geometric Exercises in Paper Folding. By T. Sundara Row. Edited and revised by W. W. Beman and D. E. Smith. With many half-tone engravings from photographs of actual exercises and a package of colored papers for folding. 8vo, cloth, x+148 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

This book contains proofs of a number of géometric theorems by means of folding paper. By this means some important geometric processes can be effected much more easily than with compasses and straight edge. While it is not possible to fold paper so as to represent a circle, yet a number of points on a circle can be obtained by this method. In this book are treated, The Square, the Equilateral Triangle, Squares and Rectangles, Pentagon, Hexagon, the Octagon, the Nonagon, the Decagon. and the Dodecagon, the Pentedecagon, Series, Polygons, General Principles, the Conic Sections, the Circle, the Ellipse, the Parabola, the Hyperbola, Miscellaneous Curves, the Cissoid, the Conchoid, the the Witch, the Cubical Parabola, the Harmonic Curve, the Ovals of Cassina, the Logarithmic Curve, the Catenary, the Cardioid, the Limaçon, the Lemniscate of Bernouilli, and the Cycloid. The work is one by which new interest may be awakened in the ever interesting and fascinating subject of Geometry. B. F. F.

CHARLES HERMITE.

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ON THE HISTORY OF SEVERAL FUNDAMENTAL THEOREMS IN THE THEORY OF GROUPS OF FINITE ORDER.*

By DR. G. A. MILLER, Leland Stanford Jr. University, Palo Alto, Cal.

The three most prominent sources of the theory of groups of finite order are: geometric transformations, theory of numbers, and theory of algebraic equations. The group properties of the totality of the rotations through a submultiple of 360°, or through 180° around three concurrent lines, each perpendicular to the plane determined by the other two, must have been observed very early. The prominent place which the five regular polyhedra—the Platonic bodies—occupy in the history of thought make it appear probable that the group properties of the totality of the rotations into themselves of these solids were observed quite early.†

The elementary theory of congruences and especially the combinatory laws of the n roots of unity when they are multiplied together exhibit group properties which cannot have escaped the notice of the early students of these questions. As Cayley aptly says (*Philosophical Magazine*, 1854, page 40) "A number of elementary group concepts have been employed by mathematicians for a long time but have not been especially noted on account of their great simplicity." Some of the theorems by Gauss and Schering along this line are of sufficient interest in themselves to attract attention. In particular the theorem by Gauss to the effect that an Abelian group can be resolved in only one way in-

*Read before the American Association for the Advancement of Science, August, 1901.

†Bravais, *Liouville's Journal*, Vol. 14, 1849, page 167.

to factor groups whose orders are prime to each other, and the theorem by Schering to the effect that every Abelian group can be resolved in more than one way into factor groups such that the order of each is divisible by the orders of all those which follow it, have a strong claim to being considered the beginning of the theory of abstract groups of finite order.

The early developments in substitution groups were made in connection with the theory of algebraic equations, starting with the theories of Lagrange and becoming prominent in the development and extension of these theories by Ruffini, Abel, and Galois.* Cauchy seems to have been the first to develop the theory of substitution groups independently of any direct application to the theory of equations, and he was also the first to prove any remarkable theorems in this theory. The earlier theorems were almost self-evident, but this cannot be said of the theorem that a group must involve a substitution of a prime order (p) whenever its order is divisible by p.† This theorem, known in the theory of groups as Cauchy's theorem, is especially interesting since it is independent of the particular notation by means of which a group is represented and hence applies directly to the theory of abstract groups.

In proving this theorem, Cauchy employed another interesting theorem, due to himself, viz., that the symmetric group of degree n contains a subgroup of order p^m, p^m being the highest power of p that divides $n!$ About thirty years later Sylow extended very materially these results due to Cauchy by proving that a group (G) of order g must always contain $1+kp$ subgroups of order p^α, p^α being the highest power of p that divides g, and that all of these subgroups are conjugate under G. This fundamental theorem is known as *Sylow's theorem.* It is sometimes called the Cauchy-Sylow theorem, and was first published in the *Mathematische Annalen*, Vol. 5.

About twenty years later Frobenius pnblished, in the *Berliner Sitzungsberichte*, a very important extension of Sylow's theorem in which it is proved that the number of subgroups of order p^α which are contained in any group is $\equiv 1 \bmod p$. These subgroups do not necessarily form a single set of conjugates unless p^α is the highest power of p that is contained in the order of the group. Frobenius deserves credit also for a simple proof of Sylow's theorem by means of a more prominent use of the concept of complete sets of conjugates. This concept is prominent in a large number of the publications by Frobenius.

From what precedes it may be observed that a very important part of Sylow's theorem was constructed on French soil. A Norwegian greatly enlarged and beautified this structure. Finally a German enhanced its usefulness by extending it very materially in certain directions. Hence this fundamental theorem stands before us as a truly international structure, which has required more than fifty years for its erection. One of its main contributors (Sylow) has published comparatively little along the line of group theory, while the other two have published extensively along this line.

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*Cauchy, ***Physique Mathematique***, Vol. 3, 1844, page 250.

†Pierpont, ***Bulletin of the American Mathematical Society***, Vol. 1, 1895, page 196.

The article in which Sylow made his famous theorem known contains another of not much less importance, viz., A group of order p^a contains at least p invariant operators. This furnishes the starting point of a large number of the theorems relating to groups whose order is a power of a prime. On account of its fundamental importance it has been proved in several different ways. The most recent of these proofs is deduced from the known fact that the subgroup which omits a given element of a transitive substitution group of order p^m omits a power of p of its elements.*

Closely related to this theorem is the question in regard to the largest Abelian subgroup in a group of order p^a. The first step towards a theorem along this line was the observation that every group of order 16 contains an Abelian subgroup of order 8. A little later it was observed (Comptes Rendus, February, 1896) that every group of order p^4 contains an Abelian subgroup of order p^3. This fact was then proved independently of the list of all the possible groups of order p^4.† Finally, it was proved in the *Messenger of Mathematics*, 1897, that every group of order p^α contains an Abelian group of order p^m, whenever $\alpha > \frac{m(m-1)}{2}$.

A more special theorem which has a somewhat singular history is the one which states that there are just fifty-one groups of order 32. In 1896 a Frenchman stated in *Comptes Rendus de l' Académie des Sciences* that he had found seventy-five groups of this order and had not yet finished his enumeration. Shortly after this an American stated in the same journal that he also had investigated this problem and that he had proved that there are only fifty-one such groups.

About two years later an Italian published in *Annali di Matematica* his investigations in regard to all the groups of order p^5 and stated therein that both of these results were incorrect inasmuch as the correct number of these groups was just 50. Very shortly after this the said American reaffirmed his former results and called attention to several errors in the enumeration of the Italian. Finally, the latter published a separate investigation of this subject in which he agreed with the American and stated that he considered the determination of all the groups of order 32 settled beyond a doubt.

In tracing the history of a concept or theorem the most difficult part is that in which the concept seems implied, for one is always in danger of reading things into a paper, which the author did not have in mind. This is perhaps especially true of the concepts of isomorphisms and group of isomorphisms. Both of these are implied in the theory of intransitive substitution groups, but it is difficult to say whether any of the earliest workers along this line had them distinctly in mind.

It appears that Hölder and Moore were the first to call explicit attention to the fact that the totality of the simple isomorphisms of a group with itself constitute a group. Hölder remarked that the group of cogredient isomorphisms

***American Journal of Mathematics*, Vol. 23, page 173.

†*Quarterly Journal of Mathematics*, Vol. 28, page 233.

is an invariant subgroup of the group of isomorphisms. Recently the group of isomorphisms of a group (G) has been studied from various standpoints. It has been observed that it is the largest subgroup of the holomorph of G, which does not include one of the elemeuts. When G is cyclic it is Abelian, but when G is any non-cyclic Abelian group its group of isomorphisms is non-Abelian. In the latter case its invariant operators are those which transform each of the operators of G into the same power.

The second volume of Weber's Algebra is one of the best and most popular works on the theory of groups of finite order. This may perhaps justify our noting a very singular error which occurs on page 54 of the first edition. The author states at this place that the natural numbers when combined by multiplication furnish the most important example of a commutative group. It is very easy to see that these numbers, combined in the said manner, do not form any group at all. So that what is called the most important example has no existence.

In his second edition Weber recognizes this fact and replaces the given example by another "most important example." It would probably be difficult to prove that numbers combined by multiplication furnish a more important example of Abelian groups than when they are combined by addition. The greatness of this work may perhaps justify the noting of another slight defect, viz., in the treatment of the new subject of commutator groups, the author gives reference to the man who first published the name of these groups but he does not give any reference to the one who first published their properties.

SUPPLEMENTARY REPORT ON NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED, University of Texas, Austin, Texas.

When at the Columbus Meeting of the American Association I had the honor of making a Report on Non-Euclidean Geometry, it was mentioned that my own Bibliography of Hyper-space and Non-Euclidean Geometry in the *American Journal of Mathematics* (1878) giving 81 authors and 174 titles, when reprinted in the collected works of Lobachevski (Kazan, 1886) gives 124 authors and 272 titles ; while Roberto Bonola had just given (1899) a Bibliography of the Foundations of Geometry in relation to Non-Euclidean Geometry containing over 350 titles with some repetitions.

Bonola in 1900 finished a second part of this Bibliography in which the single section headed "Historical, Critical, and Philosophical Writings" gives 96 authors and 150 titles. It thus becomes very evident that a most important function of your Reporter is the selection of what writings to bring forward for especial mention as of paramount importance and typical of the main stream of advance. In the Columbus Report I particularly stressed the work of two

authors whom I brought forward together and to whom I devoted about a quarter of that report.

The Report first appeared in *Science* for October 20, 1899, and you may imagine that it was reassuring when on October 22 (old style), 1900, the Commission of the Physico-Mathematical Society of Kazan found the scientific merit of the works of these authors, A. N. Whitehead and Wm. Killing, equal for the great Lobachevski prize, and had to decide between them by the drawing of lots.

In his report on the work of Whitehead, Sir Robert Ball says of the "Universal Algebra" says : "Several other writers, to whom of course Mr. Whitehead makes due acknowledgment, have approached the study of non-Euclidean Geometry by the aid of Grassmann's methods, but the systematic and most instructive development of the subject in Book VI is, I believe, new, as are also many of the results obtained.

"The superiority of Whitehead's methods appears to lie in the two following features:

"1°. That he can treat n dimensions by practically the same formulae as those used for two or three dimensions.

"In this I think he has made a considerable advance upon the methods, ingenious and beautiful as some of them no doubt are, which have been used by previous investigators.

"2°. The various kinds of space, parabolic, hyperbolic, and elliptic (of two kinds), present themselves in Whitehead's methods quite naturally in the course of the work, where they appear as the only alternatives when certain assumptions have been made.

"Moreover, the results have been obtained in such a way that it is easy for the reader to develop for one of the other spaces properties treated out in full for one space only.

"The book deserves in the highest degree the attention of the student of modern mathematical methods, and it marks so great an advance that it is, in my judgment, well worthy of the important prize in view of which this report is prepared.

"Mr. Whitehead's memoir on geodesics in Elliptic Space appears to me to indicate great power in dealing with a very difficult problem. I believe it to be of much importance, as the Geodesics in the generalized space conceptions had been but little studied."

In the corresponding Report on the work of Killing, Professor Engel, of Leipzig, says of the "Grundlagen der Geometrie": "This work is, from the first to the last page, a justification and detailed development of the circle of ideas which we are accustomed to understand under the expression non-Euclidean Geometry."

"Already so many preliminary questions have been settled," said Killing in the preface to his first volume, "that the final solution can be hoped for at a not too distant time."

"These words written in 1893," says Engel, "have meanwhile most recently (1899) found a highly striking confirmation in many directions through Hilbert's investigations.

"The geometries possible with the Euclidean, namely, the Lobachevski-Bolyaian, the Riemannian, and the elliptic, Killing developes, each for itself, in Euclidean way up to a certain grade.

"Also it should not be forgotten that Killing was the first, who (1879, *Crelles Journal*, Bd. 83) made clear the difference between the Riemannian and the elliptic space (or as he calls it, the Polar form of the Riemannian).

"The fourth section treats the Clifford-Klein space forms, in whose investigation Killing himself has taken a conspicuous part (by a work in Bd. 39 of the *Mathematische Annalen*, 1891). The great importance of these space forms rests upon this, that they show with especial clearness, what a mighty difference it makes whether we, from the beginning, assume the geometric axioms as valid for space as a whole or merely for an every way bounded piece of space. In the first case we obtain, besides the Euclidean, only the three previously mentioned non-Euclidean space forms.

"In the second case appears also a manifoldness, at present not yet dominated, of different space forms.

"The treatment of continuity and the ratio-idea in Euclid gives occasion for a nearer investigation of the so-called Archimedes' Axiom.

"Finally, as the first attempt to illuminate in conjunction all the different questions which have grouped themselves about the problem mentioned, and to collect all the means, which numerous mathematicians, and not least the author himself, have made for solving the problem, this work will for long retain its value.

"That precisely the founding of Geometry since the appearance of this book has been advanced in wholly unexpected way by Hilbert, cannot lessen Killing's merit. His work remains still by far the best means for mastering the researches which have appeared in this realm up to 1898."

These interesting extracts I take from the Russian pamphlet just issued at Kazan and furnished me by my friend Professor Vasiliev.

In his paper "Ueber Nicht-Euklidische und Linien-Geometrie" (Greifswald, 1900) Professor E. Study voices a profound truth when he says: "The conception of geometry as an experimental science is only one among many possible, and the standpoint of the empiric is as regards geometry by no means the richest in outlook. "For he will not, in his one-sidedness, justly appreciate the fact that in manifold and often surprising ways the mathematical sciences are intertwined with one another, that in truth they form an indivisible whole.

"Although it is possible and indeed highly desirable, that each separate part or theory be developed independently from the others and with the instrumentalities peculiar to it, yet whoever should disregard the manifold interdependence of the different parts, would deprive himself of one of the most powerful instruments of research.

"This truth, really self-evident yet often not taken to heart, applied to Euclidean and non-Euclidean geometry, leads to the somewhat paradoxical result, that, among conditions to a more profound understanding of even very elementary parts of the Euclidean geometry, the knowledge of Non-Euclidean Geometry cannot be dispensed with."

That the world has caught one deduction from this deep idea, is shown by the fact of the almost simultaneous appearance of two text-books, manuals for class use, to make universally attainable this necessary condition for any thorough understanding of any geometry, even the most elementary ; two intended, available popular treatises on this ever more essential Non-Euclidean Geometry.

One of these, just being issued by G. Carré et C. Naud, 3 rue Racine, Paris, is "La géométrie non Euclidienne" by P. Barbarin, professor at Bordeaux, a place made sacred for Non-Euclideans by the memory of Hoüel. How great and practical is the interest of this book can be gathered from the headings of its chapters :

I. *General and historical considerations.* How the non-Euclidean doctrine was born and gradually developed.

II. *Euclid's definitions and postulates.* Study of the rôle that they play in the principles of geometry. Simple and elementary exposé of the three geometries after the method of Saccheri.

III. *Distance as fundamental notion.* The definitions of the straight and plane according to Cauchy. The works of M. de Tilly.

IV. *General geometry in the plane and in space.* Résumé of the principal general propositions.

V. *Trigonometry.* Elementary demonstration, after Gérard and Mansion, of the formulas for triangles and quadrilaterals.

VI. *Measurement of areas and volumes.*

VII. *The contradictors of the Non-Euclidean geometry.* The principal objections made against the Non-Euclidean geometry. Answers to be made thereto.

VIII. *Physical geometry.* How we might attempt to find out if the physical world is not Euclidean ; how angles and distances could be measured with a much greater approximation, for example, angles with an error much less than $\frac{1}{100}$ of a second.

A brief article by Professor Barbarin, "On the utility of studying Non-Euclidean geometry," which appears in the May (1901) number of Professor Cristoforo Alasia's new Italian journal *Le Matematiche*, shows that Hoüel had reached the weighty insight which we have quoted from Study, namely, that knowledge of Non-Euclidean geometry is essential for any mastery of Euclidean geometry. Says Barbarin :

"I. The question of the source of the theory of parallels has been one of the most interesting scientific preoccupations of the century ; it has made to flow torrents of books, and given theme to remarkable works. Thanks to the theorems of Legendre, to the discoveries of the two Bolyai, of Lobachevski, and of

Riemann, thanks to the original researches of Beltrami, and of Sophus Lie, of Poincaré, Flye Ste. Marie, Klein, De Tilly, etc., we cannot any more be mistaken as to the true scope of the celebrated proposition which bears the name of Postulate of Euclid :

"1°. This is not in any way contained in the classic definitions of the straight and the plane ;

"2°. This is, among three hypotheses equally admissible, and which cannot all be rejected, only the most simple.

"Is it perhaps chance alone which gave to the great Greek geometer the choice of his system of geometry ? or did he perceive, at least in part, the difficulties and the greater theoretic complication of the other two ? We will never know with certainty.

"But in the presence of his work, so perfect and so rigorous, one thing, however, appears not to be doubtful : the place which he assigned to his proposition, the enunciation which he gave of it, attest that this proposition had to his eyes only the value of an hypothesis ; otherwise he would have formulated it in other terms and would have attempted to demonstrate it.

"The ideas of Lobachevski and of Riemann were diffused only very slowly. They were so, above all, thanks to the translations of Hoüel. This scientist, whose activity and power of work were prodigious, could not resist the desire to master all the European languages, with the aim of being able to read in their original text, and then make known to his contemporaries, the most celebrated mathematical works.

"He admired Lobachevski, whom he surnamed the *modern Euclid*, and in his course professed at the scientific faculty of Bordeaux, he did not let pass any occasion to put him in evidence.

"II. Hoüel was persuaded that the knowledge of the Non-Euclidean geometry is indispensable for thoroughly mastering the mechanism of the Euclidean geometry.

"Despite its paradoxical form, this idea is most just.

"General geometry or *Metageometry* contains in fact a great number of propositions common to all the systems, and which ought to be enunciated in the same terms in each of these. If the general proposition can be demonstrated in terms general for these, such should be preferred even if, to attain this, it be necessary to subject the ordinary form to some modification. To cite only one example, we take the convex quadrilateral inscribed in a circle.

"In Euclidean geometry, *the sum of two opposite angles is constant and equal to two right angles ;* in Non-Euclidean geometry *this sum is variable.*

"Notwithstanding this, the two forms may be reconciled, since in both cases *the sum of the two opposite angles equals that of the other two*, and this is sufficient for a convex quadrilateral to be inscriptible.

"Confronting the proposition with that which concerns the circumscribed quadrilateral, we put in full light a correlation which, *a priori*, ought evidently to exist. This correlation, which is the very heart of general geometry, and

which does not always appear in the ordinary geometry with the same clearness, can be utilized for finding new properties of the figures.

"Example. *Every conic is the locus of the points such that the sum of the tangents from these drawn to two circles is constant;* every conic then will also be *the curve envelope of the straights which cut two given circles under angles of which the sum is constant.* (Excellent problem for investigating directly).

"III. Is it expedient to associate the Non-Euclidean geometry with instruction, and in what measure?

"If we treat of higher instruction, with ardor we respond affirmatively.

"In the courses of higher geometry in the Universities the names of Bolyai, Lobachevski, Riemann have their assigned place, and there are still divers unexplored domains on the road which these scientists have opened.

"In so far as it refers to secondary instruction, the question is more delicate. The programmes of preparatory courses at the high schools contain all, or almost all, special mathematics and spherical geometry.

"It would not be then a great inconvenience to there make opportunely a discrete allusion to general geometry : on the contrary, the attention of the students and their critical spirit would be held awake by the necessity of investigating if such proposition which is expounded to them is of order particular or general.

"At least two indispensable conditions should be satisfied ; it is requisite :

"1°. *That in all the books put in the hands of the students, the hypothetical and wholly factitious character of the Euclidean postulate be put well into relief.*

"In my classes I have recourse with success to the simple procedure which follows and I recommend. Take the straight AB and the two equal perpendiculars AC, BD: the angles ACD, BDC are equal, and may be right, acute, or obtuse.

"But whichever be the one among these three hypotheses which we assume for this particular quadrilateral, we must conserve it for *all* the other like quadrilaterals. We choose the system of geometry in which these are right angles, and which corresponds to the Euclidean hypothesis.

"2°. That the invertibility of the postulate of Euclid be completely given up in all the demonstrations in which it can be done without, and where nevertheless it is wrongly used.

"See, for example, the theorem on the face angles of a trihedral or polyhedral angle.

"We should recognize that great advances have been made in these latter years in the sense indicated.

"If the ideas of general geometry tend to become popularized, the honor of it is due above all to the periodicals which have given their hospitality, and in special manner to *Mathesis*, so ably edited by our excellent confrère P. Mansion of Ghent.

"In the course of the last eight or ten years this journal has published numerous articles on Metageometry, written with as much competence as good sense. We counsel their perusal."

It will be seen from our quotation, that Professor Barbarin bases his exposition on the method of Saccheri as the simplest. The same is true in the other new text-book, Manning's Non-Euclidean Geometry (Boston, Ginn & Co., 1901, 8vo. pp. v—95).

Saccheri's first proposition is (THE AMERICAN MATHEMATICAL MONTHLY, June, 1894, Vol. I, p. 188): "If two equal straights AC, BD, make with the straight AB angles equal toward the same parts: I say the angles at the join CD will be mutually equal."

On the next page is "Proposition II. The quadrilateral $ABCD$ remaining the same, the sides AB, CD are bisected in points M and H. I say the angles at the join MH will be on both sides right."

Professor Manning paraphrases these two together on page 5: "If two equal lines in a plane are erected perpendicular to a given line, the line joining their extremities makes equal angles with them and is bisected at right angles by a third perpendicular erected midway between them."

Under the heading *Definitions*, Saccheri says: "Since (from our first) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we will call base), makes equal angles with these perpendiculars, three hypotheses are to be distinguished according to the species of these angles.

"And the first, indeed, I will call hypothesis of right angle; the second, however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle." This Manning paraphrases as follows, under the heading *The Three Hypothesis:* "The angles at the extremities of two equal perpendiculars are either right angles, acute angles, or obtuse angles, at least for restricted figures. "We shall distinguish the three cases by speaking of them as the hypothesis of the right angle, the hypothesis of the acute angle, and the hypothesis of the obtuse angle, respectively."

Saccheri's Proposition III is: "If two equal straights, AC, BD, stand perpendicular to any straight, AB: I say the join CD will be equal, or less, or greater than AB, according as the angles at CD are right, or obtuse, or acute."

This Manning paraphrases as follows: "The line joining the extremities of two equal perpendiculars is, at least for any restricted portion of the plane, equal to, greater than, or less than the line joining their feet in the three hypotheses respectively."

In the same way is paraphrased Saccheri's Proposition IV, the converse of III.

Saccheri's Corollary about quadrilaterals with three right angles is given by Manning on page 12.

Saccheri's Proposition V is: "The hypothesis of right angle, if even in a single case it is true, always in every case it alone is true."

In giving this, Manning has: "If the hypothesis of *a* right angle," etc., evidently a slip for his usual *the* right angle. Of course the Latin original, of which I have, so far as I know, the only copy on this continent, has no article.

Proposition VI and Proposition VII are combined by Manning on page 13.

Proposition IX is reproduced on page 14. Proposition X is given on page 9.

In Proposition XI Saccheri with the hypothesis of right angle demonstrates the celebrated Postulatum of Euclid, thus showing that his hypothesis of right angle is the ordinary Euclidean geometry.

Manning says, page 27: "The three hypotheses give rise to three systems of Geometry, which are called the Parabolic, the Hyperbolic, and the Elliptic Geometries. They are also called the Geometries of Euclid, of Lobachevski, and of Riemann."

Now Saccheri in his demonstration of Proposition XI makes, almost in the words of Archimedes, an assumption, introduced by the words, "it is manifest," which we now call, for convenience, Archimedes' Axiom.

In his futile attempts at demonstrating the parallel-postulate, Legendre set forth two theorems, called Legendre's theorems on the angle-sum in a triangle. They are

1. In a triangle the sum of three angles can never be greater than two right angles.

2. If in any triangle the sum of the three angles is equal to two right angles, so it is in every triangle.

In addition to assuming the infinity or two-sidedness of the straight, in his proofs of these theorems Legendre uses essentially the Archimedes Axiom. Thence he gets that the angle-sum in a triangle equaling two right angles is equivalent to the parallel-postulate, all of which is really what Saccheri gave a century before him, now just reproduced by Barbarin and Manning, as before by De Tilly. Even Hilbert in his "Vorlesung ueber Euklidische Geometrie" (Wintersemester 1898-99), for a chance to see Dr. von Schaper's Autographie of which I am deeply grateful to Professor Bosworth, gives the following five theorems and then says: "Finally we remark, that it seems as if each of these five theorems could serve precisely as *equivalent of the Parallel Axiom.*" They are

1. The sum of the angles of a triangle is always equal to two right angles-

2. If two parallels are cut by a third straight, then the opposite (corresponding) angles are equal.

3. Two straights, which are parallel to a third, are parallel to one another.

4. Through every point within an angle less than a straight angle, I can always draw straights which cut both sides [not perhaps their prolongations].

5. All points of a straight have from a parallel the same distance.

But since then a wonderful discovery has been made by M. Dehn.

It was known that Euclid's geometry could be built up without the Archimedes Axiom. So arises the weighty question: *In such a geometry do the Legendre theorems necessarily hold good?* In other words: Can we prove the Legendre theorems without making use of the Archimedes Axiom ?

This is the question which, at the instigation of Hilbert, was taken up by Dehn. His article was published July 10th, 1900 (*Mathematische Annalen*, 53 Band, pp. 404-439). Dehn was able to demonstrate Legendre's second theorem

without using any postulate of continuity, a remarkable advance over Saccheri, Legendre, De Tilly.

But his second result is far more remarkable, namely, that Legendre's first theorem is indemonstrable without the Archimedes Axiom.

To prove this startling position, Dehn constructs a new Non-Euclidean Geometry, which he calls a "Non-Legendrean" Geometry, in which through every point an infinity of parallels to any straight can be drawn, yet in which nevertheless the angle sum in every triangle is greater than two right angles.

Thereby is the indemonstrability of the first Legendre theorem without the help of the Archimedes Axiom proven.

Dehn then discusses the connection between the three different hypotheses relative to the sum of the angles [the three hypotheses of Saccheri, Barbarin, Manning] and the three different hypotheses relative to the number and existence of parallels. He reaches the following remarkable propositions :

From the hypothesis that through a given point we can draw an infinity of parallels to a given straight it follows, if we exclude the Archimedes Axiom, *not* that the sum of the angles of a triangle is less than two right angles, but on the contrary that this sum may be (a) greater than two right angles, (b) equal to two right angles.

The first case (a) is proven by the existence of the Non-Legendrean Geometry.

To demonstrate the second case (b), Dehn constructs a geometry wherein the parallel-axiom does not hold good, and wherein nevertheless are verified all the theorems of Euclidean geometry: the sum of the angles of a triangle is equal to two right angles, similar triangles exist, the extremities of equal perpendiculars to a straight are all situated on the same straight, etc.

As Dehn states this result : There are Non-Archimedian Geometries, in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles. Such a geometry he calls "*Semi-Euclidean.*"

Therefore it follows that none of the theorems: the angle-sum in the triangle is two right angles, the equidistantial is a straight, etc., can be considered as equivalent to the parallel-postulate, and that Euclid in setting up the parallel-postulate hit just the right assumption.

This is a marvelous triumph for Euclid.

Finally Dehn arrives at this surprising theorem : From the hypothesis that there are no parallels, it follows that the sum of the angles of a triangle is greater than two right angles.

Thus the two non-Euclidean hypotheses about parallels act in a manner absolutely different with regard to the Archimedes Axiom.

The different results obtained may now be tabulated thus :

The angle-sum in the triangle is :	Through a given point we can draw to a straight :		
	No parallel.	One parallel	An infinity of parallels
$>2R$	Elliptic Geometry	(Impossible)	Non-Legendrean Geometry
$=2R$	(Impossible)	Euclidean Geometry	Semi-Euclidean Geometry
$<2R$	(Impossible)	(Impossible)	Hyperbolic Geometry

Riemann, Helmholtz, and Sophus Lie founded geometry on an analytical basis in contradistinction to Euclid's pure synthetic method. They elected to conceive of space as a manifold of numbers. In the Columbus Report is an account of the Helmholtz-Lie investigation of the essential characteristics of space by a consideration of the movements possible therein.

This is notably simplified if we suppose given *a priori* the graphic concepts of straight and plane, and admit that movement transforms a straight or a plane into a straight or respectively a plane. Killing determines analytically the three types of projective groups, but the same results are reached in a way geometric and purely elementary by Roberto Bonola in a beautiful little article entitled "Determinazione, per via geometrica, dei tre tipi di spazio: Iperbolico, Ellittico, Parabolico" (Rendiconti del Circolo Matematico di Palermo, Tomo XV, pp. 56-65, April. 1901).

In 1833 was published in London the fourth edition of an extraordinary book (3rd Ed. 1830) by T. Perronet Thompson of Queen's College, Cambridge, with the following title :

"Geometry without Axioms.

"Being an attempt to get rid of Axioms and Postulates ; and particularly to establish the Theory of Parallel Lines without recourse to any principle not grounded on previous demonstration.

"To which is added an appendix containing notices of Methods at different times proposed for getting over the difficulty in the Twelfth Axiom of Euclid." 8vo, pp. x—148. This dissects most brilliantly twenty-one methods of getting rid of Euclid's postulate; so brilliantly that it deserves to be reprinted and could scarcely be improved upon. Then, nothing daunted by the failure of every one else of whom he has ever heard, the brave Thompson adds one of his own, which perhaps he also afterward impaled upon the point of his keen dissecting scalpel, for he lived long and prospered. In 1865 De Morgan, whose unknown letters to Sylvester I had the pleasure of publishing in the *Monist*, writes :

"With your note came an acknowledgment from General Perronet Thompson, B. A. of 1802, and Fellow of Queen's before he was an ensign. And he works at acoustics as hard as ever he did at the Corn Laws."

But even in 1833, had he but known it, the question of two thousand years, as to whether Euclid's Parallel-Axiom could be deduced, had been settled at last by the creation and indeed publication, by Bolyai and also by Lobachevski, of a geometry in which it is flatly contradicted.

The newly created methods, which thus settled this old, old question, give entirely new views concerning the investigation of axioms in general; and this diamond mine has been masterfully preempted by Hilbert of Göttingen. His wonderful "Grundlagen der Geometrie" is ablaze with gems from this Non-Euclidean mine.

After Bolyai and Lobachevski, Hilbert's closest forerunner is Friedrich Schur of Karlsruhe. One of the most fundamental advances of this decade is the strict rigorous reduction of the comparison of areas to the comparison of sects.

This was first given on January 23, 1892, by Schur before the Dorpater Naturforscher-Gesellschaft. The account printed in Russia in the society's Proceedings, a *Referat* given by Schur, is of course almost inaccessible, nor is this inaccessibility much lessened for us by the fact that it has been translated into Italian (*Per. di Mat.* VIII, page 153).

The essence of the matter is the proof that, a certain sect being taken as the measure of the area of a triangle, the *sum* of these sects is *the same* for any set of triangles into which a given polygon can be cut, and so gives a sect which may be taken as the measure of the area of the polygon. The *Referat* begins as follows: "On the surface-content of plane figures with straight boundaries, by Friedrich Schur.

"So simple a problem as the measuring of plane figures with straight boundaries, as it seems from the literature to me accessible, has not yet been set forth with the rigor and purity of Method herein possible.

"Not to mention the introduction of endless processes, still general magnitude-axioms, which are only then immediately clear when these magnitudes ara straight sects, their comparison therefore capable of being made by superposition.

"Such a general magnitude-theorem, which is used in all text-books of elementary mathematics known to me in proving the theorem of the equal area of two parallelograms with common base and equal altitude, is, *e. g.* this, that the subtraction of equal magnitudes from equal magnitudes gives again magnitudes.

"If the sides of the two parallelograms lying opposite the common base have a piece or at least a point in common, then the two parallelograms can at once be cut into parts such, that each part of the one parallelogram corresponds to a part congruent to it of the other parallelogram.

"On the contrary, if those two sides have no point in common, then it has been believed that this method of proof for the equality of area, simple and standing upon a sharp definition, must be renounced, and it has been replaced, as is known, by this, that each of the two parallelograms is represented as the difference between the same trapez and one of two congruent triangles.

"But before the measurement of plane surfaces by sects has been attained, which just first becomes possible through the theorem to be proven, the application of the above magnitude-theorem is justified by nothing.

"We must therefore throw away this method of proof, and that so much the more, as in every case each of two parallelograms with common base and equal altitude in very simple way comprehensible to every scholar can be so cut into a number of parts, that to each part of the one parallelogram corresponds a part congruent to it of the other.

"One may find that, *e. g.* set forth in Stolz's Vorlesungen ueber allgemeine Arithmetik I. Theil (Leipzig, 1885) s. 75 ff.

"We can still somewhat simplify this method, and lessen the number of parts. Draw, namely, through each of the two end-points next one another of the sides lying opposite the common base, parallels to the sides of the other parallelogram, and prolong these to the two outer of the sides not parallel to the base. The join of the two end-points so obtained is then parallel to the base, and cuts from the two parallelograms two new parallelograms which without anything further are divided into triangles every two congruent to one another.

"If then the sides opposite the common base of the remaining parallelograms again have no common point, then we proceed just so with them, and come thus, after a finite number of repetitions, to a pair of parallelograms, to which the customary procedure can be applied.

"If the distance of those two neighboring end-points of the sides opposite the base is greater than the n-fold of the base, on the other hand at the highest equal to the $(n+1)$-fold of the base, then is each parallelogram cut into a trapez (respectively triangle), three triangles and n parallelograms, and each of such parts of the one parallelogram corresponds to a part congruent to it of the other."

Now it so happens that I myself had reached this method and published it seven years before Schur in my *Elements of Geometry* (John Wiley & Sons, New York). It may be described more concisely as taking away pairs of congruent triangles each with base equal to the common base of the two parallelograms and sides respectively parallel to their other pairs of sides, until we have left two parallelograms to which the customary dissection into a triangle and trapezoid will apply, to finish with congruent parts.

But this demonstration, though the very simplest possible, yet postulates the Archimedes Axiom, though neither I myself in 1885 nor Schur seven years later in 1892 said a word about this assumption. However, before 1898 Schur became conscious that elementary geometry can be built up without the Archimedes Axiom. He states this in the preface to his remarkable *Lehrbuch der analytischen Geometrie* (Leipzig, Veit & Comp., 1898), referring to his article "Ueber den Fundamentalsatz der projectiven Geometrie" (*Matematische Annalen*, Bd. 51), where he proves the theorems of Desargues and of Pascal without using either the parsllel-postulate or the Axiom of Archimedes, proving that the ordinary sect-calculus can be built up independently of number-measure and the Archimedean postulate.

Professor Anne Bosworth of Rhode Island, has followed this up by actually constructing in her Doctor's Dissertation at Göttingen (1900) under Hilbert, a Sect-Calculus independent of the Parallel-axiom. This is a beautiful piece of Non-Euclidean Geometry, and is, so far as I know, the first feminine contribution to our fascinating subject.

In 1899 appeared Hilbert's "Grundlagen der Geometrie," in which the remarkable contributions of Schur are all retouched by a master hand.

In Schur's proof of the Pascal theorem the space axioms are used. Hilbert replaces them by the parallel-axiom, thus proving Pascal as a theorem of plane Euclidean geometry.

Schur makes a sect-calculus, and shows that the theory of proportion can be founded without the introduction of the difficult idea of the irrational number. He indicates that this can be done without the Archimedes Axiom. Hilbert actually does it.

Schur proves for the first time the fundamental theorem for a rigorous treatment of area. Hilbert simplifies this proof, and proceeds to treat this whole subject without the Archimedes Axiom, making here the new distinction between flächengleich and inhaltsgleich.

Two polygons are said have *equal surface* when they can be resolved into a finite number of triangles congruent in pairs.

Two polygons are said to have *equal content* if it is possible to add to them polygons of equal surface so that the two new compound polygons have equal surface.

Thus Euclid only tried to treat *equal content*, and Hilbert is here a return to the great Greek.

The intense interest in all these unexpected developments is voiced in a handsome volume: Questioni riguardanti la geometria elementare (Bologna, 1900, 8vo, pp. vii-532) edited by Federigo Enriques who has been chosen to contribute the part on the foundations of geometry to the great German Encyclopaedia of the Mathematical Sciences, and who contributes the first article (28 pages) to this Italian work. It is entitled: "On the scientific and didactic importance of the questions which relate to the principles of geometry."

The whole book may be properly described as an outcome of the non-Euclidean geometry, but more specifically, the longest of the fourteen articles which make it up is by Bonola: "On the theory of parallels and on the non-Euclidean geometries" (80 pages, 26 figures). The first fifty of his eighty pages are devoted to an historico-critical exposition; the last thirty to general theory, hyperbolic geometry, elliptic geometry. Though the article was published only last year, it is in certain respects antiquated. The proofs freely use the Archimedes postulate, without saying anything more about it than I did in 1885, that is, nothing at all. His §7 is headed "Postuales equivalent to the postulate of Euclid," and gives those adopted by Proclos, Wallis, Bolyai Farkas, Carnot, Legendre, Laplace, Gauss. But now we know that all these men failed in attempting to rival the choice of Euclid. Their axioms are not the equivalent of his immortal

immortal postulate. In this section the name Legendre is misspelled, and in § 5 Bonola says, "The attempts of Legendre for the demonstration of the Euclidean hypothesis, published in the various editions of the *Elements of Euclid*, which appear under his name," etc.

Of course Legendre never published any edition of Euclid. It was on the contrary Legendre's Geometry which cursed the subject with that definition, "A straight line is the shortest distance between two points," which still disgraces the beautifully illustrated book of Phillips and Fisher of Yale.

Again, in § 12 Bonola misquotes in a very important particular the title of the only thing Bolyai János ever published, his renowned Appendix, in which title, instead of "Johanne Bolyai de eadem," Bonola has "Johanne Bólyai de Bólya." Again in § 8 Bonola is still expressing the hope that the examination of the inedited manuscripts of Gauss may show some ground for the pretence that Gauss had some part, however minute, in the creations of Bolyai, Lobachevski, and Riemann. But these manuscripts have already been most sympathetically edited by Professor Paul Staeckel, their publication making a goodly quarto, in a review of which for *Science*, under the heading "Gauss and the Non-Euclidean Geometry," I find they only strengthen the already existing demonstration that neither of the creators of the Non-Euclidean Geometry owed even the minutest fraction of an idea or suggestion to Gauss.

This is re-proven by the correspondence of Gauss and Bolyai Farkas, so sumptuously published in royal quarto by the Hungarian Academy of Science, edited by Staeckel and Franz Schmidt, chiefly valuable for its references to the immortal boy Bolyai János, of whom unfortunately no portrait exists.

And now a word in conclusion.

Thinking is important for life. So much so, that evolution in thinking has dominated all other evolution. In all thinking enters a creative element. There is not any pure receptivity. Nothing can be described except in terms of a precreated theory. The business of science is the making of these theories, and the continual remaking and bettering of these theories. The higher races of mankind, and chiefly the Greeks, created and elaborated a scheme for dominating what a popular terminology calls the facts and laws presented by the spatial relations of things.

The scheme was only one of an indefinite number of possible schemes, but as coördinated and systematized by a great constructive genius, Euclid, the first professor of mathematics at the University of Alexandria, it proved so efficient, so effective for life, that all educated men accepted it as part of their common equipment.

Though it promises no heaven, though it threatens no hell, though it mentions no angels, no devils, yet Euclid's Elements of Geometry, simply as conveying a necessary instrument for the conduct of civilized life, has appeared in more than one thousand four hundred different editions. [Prof. Riccardi : Saggio di una bibliografia euclidea (Bologna, *Memorie* (5), I, 1890)].

Euclid gave to educated mankind a common language for description of

the spatial, a common mental basis for thought about extension. Euclid's geometry is a certain theory for a specific natural science, a mental construction to explain, to master, to communicate or transmit, and to prophesy certain physical phenomena, the spatial or extensive phenomena. Therefore the body of its doctrine is a system of theorems deduced in logical way from certain unproven and in part absolutely and finally indemonstrable assumptions. Such a one is the world-renowned parallel-postulate, which is absolutely incapable of being proved in any way whatsoever, mental or physical, speculative or experimental, deductive or inductive. Therefore to substitute for it a contradiction of it, in Euclid's scheme of fundamental assumptions, is to get with certainty another equally logical theory to do all that Euclid's geometry has ever done. Of such systems each may throw light on the other, each may possess special advantages for particular applications.

But more than that: three such systems used simultaneously may be able to accomplish what no one of them could do. This is beautifully illustrated in a theory communicated to me by F. W. Frankland, using a cosmic medium in which small regions of elliptic and hyperbolic space alternate, given a strain toward parabolic space which produces an elasticity or resilience simultating the properties with which physicists have endowed their hypothetical ether.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Stock bought $m=10\%$ above par, pays $p=8\%$ on the investment. What per cent. will it pay if bought at $n=10\%$ discount?

Solution by JAMES F. LAWRENCE, A. B., Instructor in Mathematics, Rogers Academy, Rogers, Ark.

1. 100%=par value of stock$=r$.
2. $r+m=110\%$=purchase price of stock.
3. $p=8\%$=income on investment.
4. $\therefore \frac{p}{v}(v+m)=\frac{8}{100}$ of $110=8.8\%$=income on par value.
5. $v-m=90\%$=supposed purchase price of stock.
6. $v+\frac{p}{v}(v+m)=108\frac{8}{10}\%$.
7. $v+\frac{p}{v}(v+m)-(v-n)=\frac{p}{v}(v+m)+n$=income.

8. $\therefore [\frac{p}{v}(v+m)+n]\div(v-n)=\frac{p(v+m)+nv}{v(v-n)}=20\frac{8}{9}\%$.

Also solved by *M. E. GRABER.*

ALGEBRA.

125. Proposed by LESLIE L. LOCKE, Instructor in Mathematics, Michigan Agricultural College, Ingram County, Mich.

What special expedient will solve the system

$$\left.\begin{array}{l} x^4-y^4=369\ldots.(1) \\ x-y=1\ \ldots(2) \end{array}\right\}?$$

Solution by O. S. WESTCOTT, Chicago, Ill.; J. SCHEFFER, A. M., Hagerstown, Md.; J. K. ELWOOD, A. M., Pittsburg, Pa.; J. M. BOORMAN, Woodmere, N. Y.; and the PROPOSER.

From (2), $y=x-1$. Substituting this value of y in (1), $x^4-(x-1)^4=369$ or $0.x^4+4x^3-6x^2+4x=370$. Since the coefficient of x^4 is 0, therefore, if we regard the equation as of the fourth degree, one root is ∞. Hence if $x=\infty$, $y=\infty$. The remaining three values of x are found from the equation $2x^3-3x^2+2x-185=0$. This may be solved by the method of Tartaglia; or, by trial if 5 is substituted for x, the first member vanishes. Hence, $x=5$. Dividing $2x^3-3x^2+2x-185=0$, by $x-5$, we have $2x^2+7x+37=0$. $\therefore x=\frac{1}{4}[7\pm\sqrt{(-247)}]$.

$\therefore$ The values of x are ∞, 5, and $\frac{1}{4}[7\pm\sqrt{(-247)}]$, and the values of y are ∞, 4, and $\frac{1}{4}[3\pm\sqrt{(-247)}]$.

REMARK. The Proposer says his object in proposing this problem was, 1. To learn if there is a general method of solving such problems when factors can not be readily found, and, 2. To call attention to a fact that is not mentioned in many elementary text-books on Algebra, viz., the loss of a root by dividing one equation by another, or by subtracting one from another if thereby the degree of the equation is diminished.

The above contributors used some modifications in deriving the various steps in the solution ofth is problem, but these modifications were not considered of sufficient importance to warrant separate entries. Solutions were also received from P. S. Berg, G. B. M. Zerr, and H. C. Whitaker. ED. F.

126. Proposed by CHARLES C. CROSS, Meredithville, Va.

A and B run a race; B, who runs slower than A by a miles in b hours, starts first by c minutes, and they get to the n-mile stone together. Required their rates of running. If $a=1$, $b=2$, $c=2$, and $n=4$, what is the result?

Solution by GRANTLAND MURRAY, Adjunt Professor of Mathematics, Emory College, Oxford, Ga.; J. SCHEFFER, A. M., Hagerstown, Md.; H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; C. E. ARMENTROUT, A. B., Professor of Latin and Mathematics, Rockingham Military Institute. Mt. Crawford, Va.; P. S. BERG, Principal of Schools, Larimore, N. D.; M. A. GRUBER, A. M., War Department, Washington, D. C.; C. ARTHUR LINDEMANN. A. M., Professor of Mathematics and Science, Virginia Union University, Richmond, Va.; and D. B. NORTHRUP, Mandana, N. Y.

Let x=number of miles A runs per hour, then $x-a/b$=number of miles B runs per hour.

n/x=number of hours A requires to run n miles, $n/(x-a/b)$=number of B requires to run n miles.

$\therefore \frac{n}{x-a/b}-\frac{n}{x}=\frac{c}{60}$, or $bcx^2-acx-60an=0$; whence

$$x=\frac{ac\pm\sqrt{(a^2c^2+240abcn)}}{2bc}.$$

Substituting the proposed values for a, b, c, and n gives, $x=8$, and $x-a/b=7\frac{1}{2}$.

Also solved by *G. B. M. ZERR.*

GEOMETRY.

153. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If P, P', Q, Q' be the extremities of two chords of a conic section, and both chords pass through the point A, show that the sum of the squares of the reciprocals of AP, AP', AQ, AQ' is constant.

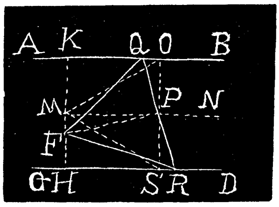
No solution of this problem has been received.

156. Proposed by F. M. McGAW, A. M., Professor or Mathematics, Bordentown Military Institute, Bordentown, N. J.

To construct an equilateral triangle such that its vertices shall be in each of two parallel lines and a point fixed between these lines.

Solution by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, High School, Manchester, N. H.

Let AB and CD be the two parallel lines, and F the fixed point between them. Through F draw HK perpendicular to CD. Make $\angle NMO=30°$. Draw MN the perpendicular bisector of HK. Draw OS perpendicular to CD. Join F and P, and through P draw QR perpendicular to FP. Join QF and RF, then FQR is the required triangle.



Proof. Triangles QOP and MFP are right triangles. $\angle QPO=\angle MPF$, being complements of the same $\angle QPM$.

$\therefore$ these triangles are similar. $\therefore$ $OP : MP :: QP : FP$, or by alternation $OP : QP :: MP : FP$. But these are homologous sides of the triangles OPM and QPF also.

$\therefore$ these triangles are similar, since they are right triangles and the legs proportional. But the $\angle OMP$ is 30° and $\angle MOP$ is 60°.

$\therefore$ $\angle QFP$ is 30° and $\angle FQP$ is 60°. Triangle FPR is easily shown to be equal to triangle FPQ.

$\therefore$ $\angle FRP=60°$. $\therefore$ triangle FQR is equiangular and therefore equilateral.

Excellent solutions were received from G. M. M. Zerr, H. C. Whitaker, J. Scheffer, and Theodore Linquist. Professors Zerr's and Whitaker's solutions were by analytical geometry; Professor Scheffer's solution was by trigonometry and the application of algebra to geometry; and Professor Linquist, of the Kansas Agricultural College, gave a very good construction by pure geometry.

157. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the locus of the center of a circle touching a given line and always passing through a given point.

I. Solution by **J. SCHEFFER, A. M.**, Hagerstown, Md.; **ELMER SCHUYLER, M. Sc.**, Reading, Pa.; **LON C. WALKER**, Palo Alto, Cal.; **H. C. WHITAKER, Ph. D.**, Philadelphia, Pa.; and the **PROPOSER**.

Take the given line as the axis of x, the line through the given point and at right angles to the given line as the y-axis, and denote the given point as $(0, y_1)$.

The required circle being of the form $x^2+y^2+2gx+2fy+c=0\ldots.(1)$, and touching $y=0\ldots.(2)$, $x^2+2gx+c=0\ldots.(3)$, and $c=g^2\ldots.(4)$.

Also, passing through $(0, y_1)$, $y_1^2+2fy_1+c=0\ldots.(5)$, and this with (4) gives $2f=-\dfrac{g^2+y_1^2}{y_1}\ldots.(6)$.

(1) now is $x^2+y^2+2gx-\dfrac{g^2+y_1^2}{y_1}y+g^2=0\ldots.(7)$.

If (x', y') be the center of (7), $x'=-g$, $y'=\dfrac{g^2+y_1^2}{2y_1}$, and eliminating g from these two equations, $x'^2=2y_1(y'-\frac{1}{2}y_1)$, a common parabola.

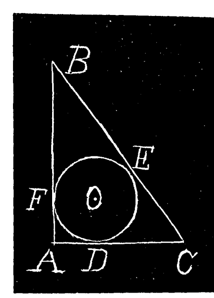
II. Solution by **G. B. M. ZERR, A. M., Ph. D.**, Philadelphia, Pa.; **P. S. BERG, B. Sc.**, Larimore, N. D.; and **H. R. HIGLEY**, East Stroudsburg, Pa.

Since the distance of the center from the given straight line is always equal to its distance from the given point, both being equal to the radius of the circle, the locus of the center is a parabola having the given straight line for directrix and the given point for focus.

ANOTHER PROOF OF THE PYTHAGOREAN THEOREM.

By **E. S. LOOMIS, Ph. D.**, Teacher of Mathematics, West High School, Cleveland, Ohio.

Let ABC be a right triangle whose sides are tangent to the circle O. Since $CD=CF$, $BF=BE$, and $AE=AD=r=$radius of circle, it is easily shown that $(CB=a)+2r=(AC+AB=b+c)$. And if $a+2r=b+c\ldots.(1)$, then $(1)^2=(2)$ $a^2+4ar+4r^2=b^2+2bc+c^2$. Now if $4ar+4r^2=2bc$, then $a^2=b^2+c^2$. But $4ar+4r^2$ is greater than, equal to, or less than $2bc$.



If $4ar+4r^2>$ or $<2bc$, then $a^2+4ar+4r^2>$ or $<b^2+2bc+c^2$; *i. e.* $a+2r<$ or $>b+c$, which is absurd.

$\therefore$ $4ar+4r^2=2bc$.

$\therefore$ $a^2=b^2+c^2$. Q. E. D.

NOTE. So far as we know, this proof has not been given before. If it has not been published before, it may be properly called a *new proof*. Dr. Loomis asks if any one can derive, by this method, a direct proof—the one above being indirect. ED. F.

CALCULUS.

116. Proposed by **JOHN M. COLAW, A. M.**, Monterey, Va.

"Prove that the length of the *greatest* beam of square section that can be cut from a log l feet long and in the shape of a conic frustum, diameters D and d, is $\frac{1}{3}lD\div(D-d)$ feet.'

Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, Ohio, and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

The altitude of the cone of which the given frustum is a part is easily found to be $Dl\div(D-d)$.

Let x=length of the required beam; then the diameter of the circle circumscribing the end of the beam is found to be $\dfrac{Dl-(D-d)x}{l}$, and the area of the end or section$=\dfrac{[Dl-(D-d)x]^2}{2l^2}$.

$\therefore$ the volume of the beam, $V=\dfrac{[Dl-(D-d)x]^2x}{2l^2}$.

Placing the first derivative of V equal to zero, and solving for x we have

$$x=\tfrac{1}{3}lD\div(D-d), \text{ or } lD\div(D-d).$$

The first of these values of x renders V a maximum, which was to be proved.

Also solved by *J. SCHEFFER*, and *W. O. PRUITT*.

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A frustum of a paraboloid of revolution closed at both ends has a given volume. Find its interior dimensions when its surface is a minimum.

Solution by the PROPOSER.

Let $y^2=4ax$ be the equation to the parabola, c=capacity of frustum.

$\therefore \frac{8}{3}\pi\sqrt{a}[(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}]+4\pi a(x_2+x_1)=u$=minimum.

$2\pi a(x_2^2-x_1^2)=c\ldots.(2)$.

$\therefore \dfrac{dx_2}{dx_1}=\dfrac{(x_1+a)^{\frac{1}{2}}-\sqrt{a}}{(x_2+a)^{\frac{1}{2}}+\sqrt{a}}\ldots.(3)$, from (1).

$\dfrac{dx_2}{dx_1}=\dfrac{x_1}{x_2}\ldots.(4)$, from (2).

$$\frac{dx_2}{da}=-\frac{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}+3a(x_2+a)^{\frac{1}{2}}-3a(x_1+a)^{\frac{1}{2}}+3\sqrt{a}(x_2+x_1)}{3a(x_2+a)^{\frac{1}{2}}+3a\sqrt{a}}\ldots(5).$$

$\dfrac{dx_2}{da}=-\dfrac{x_2^2-x_1^2}{2ax_2}\ldots.(6)$.

From (3) and (4), $x_1=\dfrac{x_2\{x_2-2a-2\sqrt{[(x_2+a]a\}}}{[\sqrt{(x_2+a)}+\sqrt{a}]^2}$, or $x_1=0$.

Eliminating dx_2/da between (5) and (6) and substituting $x_1=0$, we get

$$(8a-x_2)\sqrt{(x_2+a)}=(8a+3x_2)\sqrt{a}.$$

$\therefore x_2=24a\ldots.(7)$. From (2) and (7),

$$x_2=\left(\frac{12c}{\pi}\right)^{\frac{1}{3}},\quad a=\frac{1}{24}\left(\frac{12c}{\pi}\right)^{\frac{1}{3}}.$$

Also solved by *J. SCHEFFER*.

118. Proposed by J. W. YOUNG, Oliver Graduate Student, Cornell University, Ithaca, N. Y.

Find the differential equations of the system of parabolas, $y^2=4a^2(x+a^2)$, and of its orthogonal trajectories, and interpret the result. Find also the equation of the system of trajectories.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Since it makes no difference whether we have a^2 or a, we use the equation $y^2=4a(x+a)$. The equation of the orthogonal trajectory is, since $dy/dx=2a/y$, $1+\frac{2a}{y}.\frac{dy}{dx}=0$. Since a is variable, we must eliminate it. From the equation of the curve we find $a=-\frac{1}{2}x\pm\frac{1}{2}\surd(x^2+y^2)$, and substituting this, we have $ydx-xdy\pm dy\surd(x^2+y^2)=0$.

This being a homogeneous equation, we put $x=vy$, and since $dx=vdy+ydv$, we have $\pm\frac{dv}{\surd(1+v^2)}+\frac{dy}{y}=0$.

$\therefore\ \pm\log[v+\surd(1+v^2)]+\log y=\text{constant}$, or, taking the upper sign, $y[v+\surd(1+v^2)]=c$, or $x+\surd(x^2+y^2)=c$.

Transposing x and squaring we find $y^2=c(c-2x)$. If we take the lower sign, we find $y^2=c(c+2x)$. In either case the equation is that of a parabola of the same form as the given one. The system of the trajectories is self-orthagonal. In fact, it can be easily shown that a system of confocal conics is self-orthogonal.

Also solved by *G. B. M. ZERR*.

MECHANICS.

122. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that pressure P applied uniformly to a solid in all directions will reduce its dimensions along three perpendicular axes in ratio $1:1+p-2q$, p being the elongation along one face and q the contraction along the other. [Barker's *Physics*.]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let E=Young's Modulus, η=the ratio of the contractions (parallel to two axes) to the elongation (parallel to the third axis), e=elongation, l=reduced length of one of the perpendicular axes. Then $l(1+e)$ is the original length.

But e is composed of first the elongation parallel to the one axis$=P/E$, and the contraction parallel to the other two axes, each$=\eta P/E$.

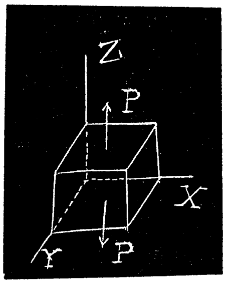
$\therefore\ e=(1-2\eta)\ P/E=p-2q$ in the problem.

$\therefore\ l:l(1+e)=l:l(1+p-2q)=1:1+p-2q$.

(See Analysis of Strains and Stresses, *Minchin's Statics*.)

II. Solution by LON. C. WALKER, A. M.

Let the figure represent a cube whose edge is unity. If a longitudinal stress P be applied along the z-axis let the increase in length be p and the accompanying decrease along both the x-axis and the y-axis be q. The dimensions of the solid would now be $1+p$, $1-q$, and $1-q$.



Now apply three longitudinal stresses of compression, one along each of the x, y, z axes. The dimensions of the solid will now become

$$1-p+2q,\ 1-p+2q, \text{ and } 1-p+2q\ldots\ (A).$$

These three equal longitudinal stresses constitute a simple hydrostatic stress. Apply a stress of compression along one axis, and a stress of tension along an axis at right angles. Then the dimensions of the solid become $1+p+q$, $1-p-q$, and $1+p-q$.

Such a stress is called shearing stress, and the strain is$=\dfrac{2(p+q)}{1}$.

From the definition of Moduli we have—

(1). Bulk Modulus$=k=\dfrac{P/1}{3(p-2q)/1}=\dfrac{P}{3(p-2q)}$.

(2). Rigidity Modulus$=n=\dfrac{P/1}{2(p+q)/1}=\dfrac{P}{2(p+q)}$.

(3). Young's Modulus$=M=\dfrac{P/1}{p/1}=\dfrac{P}{p}$.

Now if p and q be eliminated from these three equations, there results $M=\dfrac{9kn}{n+3k}$.

From (A), if the dimensions of the cube decrease to $1-p+2q$, the ratio of decrease is $1:1+p-2q$.

[See the subject "Young's Modulus" in Dr. P. G. Tait's *Properties of Matter*, and the sections of reference in the same book.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A commission merchant sold \$ W,=\$4750 worth of wheat. After deducting his commission at $m\%$,=3%, purchased with the proceeds a draft at d,=60 days at $r\%$,=10%, interest, and at $p\%$,=$\frac{3}{4}\%$ premium. What was the face of the draft?

151. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A merchant marked a lot of goods $m\%=20\%$, above cost; but, in consequence of a rise in the market price, he marked up the goods $n\%=10\%$, on the marked price. What per cent. was the last selling price of the goods? What would be his gain on sales amounting to $\$S=\$5780.50$?

ALGEBRA.

147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Prove that $x=a^x$ has never more than two real roots, and find the condition for no real roots.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\dfrac{x}{a+\alpha}+\dfrac{y}{b+\beta}+\dfrac{z}{c+\gamma}=1$, $\dfrac{x}{a+\beta}+\dfrac{y}{b+\beta}+\dfrac{z}{c+\beta}=1$, $\dfrac{x}{a+\gamma}+\dfrac{y}{b+\gamma}+\dfrac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

149. Proposed by JOSEPH V. COLLINS, Ph. D., Stevens Point, Wis.

1. How many different football elevens can be sent out from a school having twenty players? In how many ways can eleven men line up?

2. How many sets of officers (president, vice president, treasurer, and secretary) can a society of forty persons elect? How many committees of four persons, supposing no attention is paid to positions on the committees? How many committees in which the chairman is selected?

GEOMETRY.

177. Proposed by GEORGE LILLEY, Ph. D., LL. D., University of Oregon, Eugene, Ore.

If two medians of a triangle intersect each other at right angles, the third median will be the hypotenuse of a right triangle, of which the other two will be the sides.

178. Proposed by JOHN M. ARNOLD, Crompton, R. I.

A cylinder thirty feet long and two feet in diameter is to be placed in a machinery car, the inside dimensions of which are eight feet wide and eight feet high. Find length of the shortest car that will contain it.

179. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

Of all isosceles triangles inscribed in a circle, the equilateral is the maximum and has the maximum perimeter. Prove geometrically.

CALCULUS.

118. Proposed by C. C. BEBOUT, Professor of Mathematics, Elgin High School, Elgin, Ill.

A pole two inches in diameter is set vertically in a level plat of ground. At a point ten feet from the ground a string is attached. A man holds the other end of the string and walks about the pole keeping the string stretched taut, and his hand at a constant distance of four feet from the ground, till the string is all wound upon the pole. If string is ten feet long, how far has his hand moved in the operation ?

119. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics.Drury College. Springfield, Mo.

The curve $r^n = a^n \sin n\theta$ rolls along a straight line. Show that the intrinsic equation to the evolute of the locus of the pole is $s^n = a^m(1+1/n)^n \sin\psi$. [Edward's *Differential Calculus*, page 502.]

MECHANICS.

130. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Two particles are projected from A and B on the same level at α, β to horizon, and in vertical planes with which AB makes angles θ, φ. They meet and coalesce into a single particle. Find the height of the latus rectum of the subsequent path above the level of A and B.

131. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If the distributed weight on the foundations of a building is Wlb./(feet)^2, the foundations must be sunk $D = (W/w)\tan^4(\frac{1}{4}\pi - \frac{1}{2}\psi)$ feet deep in earth of density $w$lb./(feet)^3 and angle of repose ψ.

132. Proposed by THOS. U. TAYLOR, C.E., Department of Engineering, University of Texas, Austin, Tex.

1. A parabola, whose axis is vertical, is described on the vertical face of a reservoir wall. If the vertex O of the parabola is at the bottom of the wall, and the parabola intersects the surface in the points A, B, find the depth of the center of pressure of the water on the parabolic area ABO.

2. In the same problem find the center of pressure on the area included between the horizontal line through O, a vertical through B, and the curve OB.

133. Proposed by J. C. CORBIN, Superintendent of Schools, Pine Bluff, Ark.

A stick of square edged timber is 20 feet long, 10 inches square at large end, and 6 inches square at small end. How far from either end must a hand spike be placed, so that two men with the hand spike and one man at the end shall each have an equal weight to carry.

DIOPHANTINE ANALYSIS.

93. Proposed by the late SYLVESTER ROBBINS.

Solve and set forth twenty terms in some infinite series of rational parallelopipeds following the solid whose edges are 2, 3, 6, and diagonal 7.

94. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Show that the area of a rational triangle cannot be a square number.

AVERAGE AND PROBABILITY.

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A straight line is drawn at random parallel to the base of a given triangle. Three random points are then taken, one on each side of the random line and one anywhere in the triangle. Find the average area of the triangle formed by the three random points.

118. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Find the mean distance between two points taken at random in an equilateral triangle.

MISCELLANEOUS.

118. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Show how to determine the illumination at any point of the surface of the water at the bottom of a deep well, due to the light from the sky.

119. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Prove $\Sigma\cos^4x-2\Pi\cos^2x+2\Pi\sin^2x=1-\sin(\Sigma)\sin\Pi(y+z-x)$.

NOTES.

Through the kindness of Dr. D. E. Smith, we are enabled to furnish a picture of M. Hermite.

Professor W. H. Metzler, of Syracuse University, has been elected Corresponding Member of the Royal Society of Canada.

During the recent Summer Quarter of the University of Chicago there were offered fourteen mathematical courses with a total registration of three hundred seventeen.

Professor E. Woelffing, of Stuttgart, Germany, is preparing a catalogue of non-periodical literature in mathematics and mechanics, soon to be ready for publication. It will contain about sixteen thousand titles arranged under four hundred headings.

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CERTAIN HYPERBOLIC CURVES OF THE nth ORDER.*

By ALWYN CHARLES SMITH, M. S., Teacher in East Denver High School, Denver, Col.

1. In a plane are given n straight lines with the Cartesian equations

$$\begin{aligned} y&=\alpha_1x+b_1, \\ y&=\alpha_2x+b_2, \\ &\dots\dots\dots\dots \\ y&=\alpha_nx+b_n. \end{aligned}$$

A straight line with the equation

$$y=\lambda x$$

passes through the origin of coördinates and intersects the n lines in n points. In this manner n segments, measured from the origin, are obtained for each position of this variable ray through the origin. The algebraic sum of these n segments taken on this same line will determine a point, P, which will describe a curve of the nth order as λ varies from $+\infty$ to $-\infty$.

The point of intersection of any one of the given lines with the variable ray is

$$x=\frac{b_n}{\lambda-\alpha_n}, \quad y=\frac{b_n\lambda}{\lambda-\alpha_n};$$

*Summary of a thesis submitted to the Graduate Faculty of the University of Colorado for the degree of M. S., June, 1901.

therefore the coördinates (ξ, η), of the point P, are

$$\xi = \sum_1^n \frac{b_n}{\lambda - \alpha_n},$$

$$\eta = \sum_1^n \frac{b_n \lambda}{\lambda - \alpha_n}; \text{ or}$$

$$\xi = \frac{b_1}{\lambda - \alpha_1} + \frac{b_2}{\lambda - \alpha_2} + \ldots + \frac{b_n}{\lambda - \alpha_n},$$

$$\eta = \frac{b_1 \lambda}{\lambda - \alpha_1} + \frac{b_2 \lambda}{\lambda - \alpha_2} + \ldots + \frac{b_n \lambda}{\lambda - \alpha_n} = \lambda \xi.$$

From the last equation

$$\lambda = \frac{\eta}{\xi}.$$

Substituting this value of λ in the expression for ξ and simplifying the result is

$$\begin{aligned}\prod_1^n (\eta - \xi\alpha_n) = {} & b_1(\eta - \xi\alpha_2)(\eta - \xi\alpha_3) \ldots (\eta - \xi\alpha_n) + \\ & b_2(\eta - \xi\alpha_1)(\eta - \xi\alpha_3) \ldots (\eta - \xi\alpha_n) + \\ & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & b_n(\eta - \xi\alpha_1)(\eta - \xi\alpha_2) \ldots (\eta - \xi\alpha_{n-1}).\end{aligned}$$

This equation is of the nth order, therefore the locus of the point P is a curve of the nth order.

Note that the point $(\xi = 0, \eta = 0)$ satisfies the equation and is consequently a point of the locus. Mention is here made of this fact because in a large number of cases the point (0, 0) can be shown to be an isolated point.* The origin is an $(n-1)$-fold point.

The five following equations represent the locus for one, two, three, four, and five lines, respectively :

(1) $\eta - \xi\alpha_1 - b_1 = 0,$

(2) $\eta^2 - \eta\xi(\alpha_1 + \alpha_2) + \xi^2\alpha_1\alpha_2 + \xi(b_1\alpha_2 + b_2\alpha_1) - \eta(b_1 + b_2) = 0,$

(3) $\eta^3 - \eta^2\xi\Sigma\alpha_1 + \eta\xi^2\Sigma\alpha_1\alpha_2 - \xi^3\alpha_1\alpha_2\alpha_3 - \eta^2\Sigma b_1 +$
$\eta\xi[b_1(\alpha_2 + \alpha_3) + b_2(\alpha_1 + \alpha_3) + b_3(\alpha_1 + \alpha_2)]$
$- \xi^2(b_1\alpha_2\alpha_3 + b_2\alpha_1\alpha_3 + b_3\alpha_1\alpha_2) = 0,$

*Clebsch, *Vorlesungen ueber Geometrie*, Vol. I., pages 319-321.

$$(4)\ \eta^4-\eta^2\xi\Sigma\alpha_1+\eta^2\xi^2\Sigma\alpha_1\alpha_2-\eta\xi^3\Sigma\alpha_1\alpha_2\alpha_3+\xi^4\alpha_1\alpha_2\alpha_3\alpha_4-\eta^3\Sigma b_1+$$
$$\eta^2\xi\{b_1(\alpha_2+\alpha_3+\alpha_4)+b_2(\alpha_1+\alpha_3+\alpha_4)+b_3(\alpha_1+\alpha_2+\alpha_4)$$
$$+b_4(\alpha_1+\alpha_2+\alpha_3)\}-$$
$$\eta\xi^2\{b_1(\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4)+b_2(\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_3\alpha_4)$$
$$+b_3(\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_2\alpha_4)+b_4(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\}+$$
$$\xi^3(b_1\alpha_2\alpha_3\alpha_4+b_2\alpha_1\alpha_3\alpha_4+b_3\alpha_1\alpha_2\alpha_4+b_4\alpha_1\alpha_2\alpha_3)=0,$$

$$(5)\ \eta^5-\eta^4\xi\Sigma\alpha_1+\eta^3\xi^2\Sigma\alpha_1\alpha_2-\eta^2\xi^3\Sigma\alpha_1\alpha_2\alpha_3+\eta\xi^4\Sigma\alpha_1\alpha_2\alpha_3\alpha_4$$
$$-\xi^5\alpha_1\alpha_2\alpha_3\alpha_4\alpha_6-\eta^4\Sigma b_1+\eta^3\xi\Sigma b_1\alpha_2-\eta^2\xi^2\Sigma b_1\alpha_2\alpha_3$$
$$+\eta\xi^3\Sigma b_1\alpha_2\alpha_3\alpha_4-\xi^4\Sigma b_1\alpha_2\alpha_3\alpha_4\alpha_5=0.$$

By analogy it is easy to see that the general equation assumes the form

$$\eta^n-\eta^{n-1}\xi\Sigma\alpha_1+\eta^{n-2}\xi^2\Sigma\alpha_1\alpha_2-\ldots\ldots\pm\xi^n\alpha_1\alpha_2\ldots\ldots\alpha_n-$$
$$\eta^{n-1}\Sigma b_1+\eta^{n-2}\xi\Sigma b_1\alpha_2-\eta^{n-3}\xi^2\Sigma b_1\alpha_2\alpha_3+\ldots\pm\xi^{n-1}\Sigma b_1\alpha_2\alpha_3\ldots\alpha_n=0.$$

The symbolic forms

$$\Sigma b_1\alpha_2,$$
$$\Sigma b_1\alpha_2\alpha_3,$$
$$\vdots$$
$$\Sigma b_1\alpha_2\alpha_3\ldots\ldots\alpha_n$$

will be understood by comparing equations (3), (4), and (5).

2. Special applications and transformations of the above equations will be given in subsequent paragraphs.

If the n lines are parallel

$$\alpha_1=\alpha_2=\alpha_3=\ldots\ldots=\alpha_n,$$

and the resulting locus is similar to (1); and if the n lines consist of two, three, four and five sets of parallel lines the resulting loci will be similar to equations (2), (3), (4), and (5). In general the loci will differ only in having the constants

$$b_1,\ b_2,\ \ldots\ldots b_k$$

replaced respectively by the sum of the intercepts of the lines having the corresponding slopes,

$$\alpha_1,\ \alpha_2,\ \ldots\ldots\alpha_k.$$

The k systems will thus give the same locus as would the k lines which would result by considering each system separately.

If $n=2m$, *i. e.* if n is even, and the first half of the lines have the same slope as the second half then the expressions for ξ and η can be expressed by one-half the number of terms and the locus will be of the mth degree. If the lines form a regular polygon that is symmetrical with respect to the origin, n still being even, then the intercepts will be equal in pairs but of opposite sign ; consequently $\xi=0$ and $\eta=0$, and the locus is the point (0, 0). The same conclusions can be obtained from the general equation and also by plotting.

Equation (2) is a hyperbola and if we assume

$$\alpha_1=-\alpha_2, \text{ and } b_1=b_2=1,$$

it becomes

$$(6)\ \eta^2-\xi^2-2\eta=0,$$

an equilateral hyperbola.

In equation (3) assume

$$\alpha_1=-\alpha_2=-\sqrt{3},\ \ \alpha_3=0,$$
$$b_1=b_2=-2b3=2,$$

the three lines then form an equilateral triangle and the locus is

$$(7)\ \eta^3-3(\xi^2\eta+\xi^2+\eta^2)=0, \text{ (see Fig. 13).}$$

In equation (5) assume

$$\begin{aligned}
&\alpha_1=-\alpha_4=\tan 72^\circ,\\
&\alpha_2=-\alpha_3=\tan 144^\circ,\\
&\alpha_5=0,\\
&b_1=b_4,\\
&b_2=b_3,\\
&b_5=b_5, \text{ identical.}
\end{aligned}$$

The five lines then form a regular pentagon and the origin at the center of gravity ; the locus is

$$\eta^5-\eta^3\xi^2(\alpha_1^2+\alpha_2^2)+\eta\xi^4\alpha_1^2\alpha_2^2-\eta^4(2b_1+b_5+2b_5)+$$
$$\eta^2\xi^2\{2b_1\alpha_2^2+b_5-(\alpha_1^2+\alpha_2^2)+2b_2\alpha_1^2\}-\xi^4b_5\alpha_1^2\alpha_2^2=0, \text{ (see Fig. 18).}$$

3. The equation of the tangent to the curve

$$f(\xi,\ \eta)=0$$

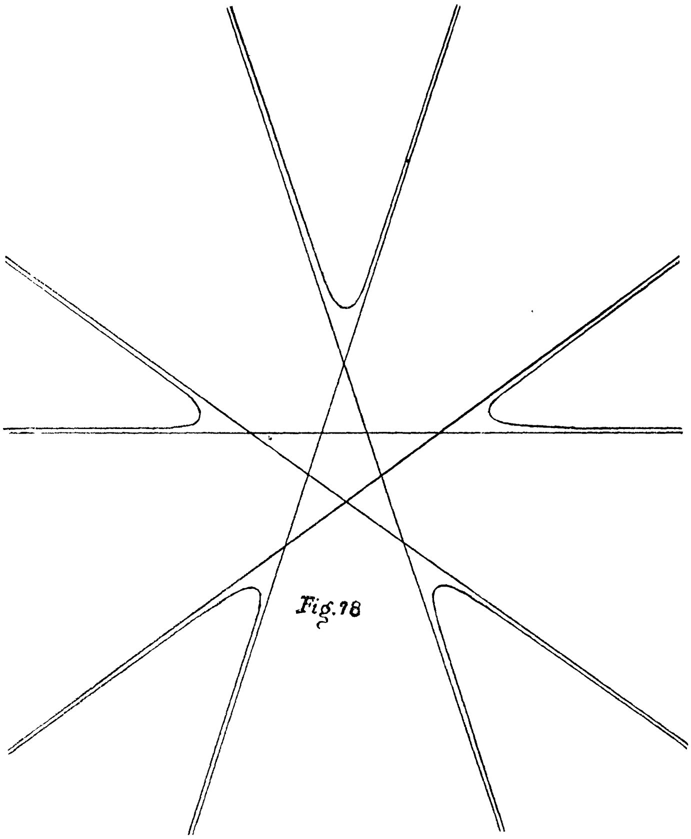

Fig. 78

is $(x-\xi)\frac{df}{d\xi}+(y-\eta)\frac{df}{d\eta}=0,$

$$\text{or } y=\frac{-(x-\xi)\frac{df}{d\xi}}{\frac{df}{d\eta}}+\eta.$$

For equation (7) the tangent is

$$(8)\quad y=\frac{2\xi(1+\eta)}{-\xi^2+\eta^2-2\eta}x+\frac{2\xi^2(1+\eta)}{-\xi^2+\eta^2-2\eta}+\eta.$$

The equations of the locus higher than the second degree

$$\frac{df}{d\xi} \text{ and } \frac{df}{d\eta}$$

will contain no terms independent of ξ or η and consequently the value of y in the equation of the tangent for the point (0, 0) is indeterminate. Equation (8) will serve as an application. This is a further illustration that the origin is often an isolated point.

The general method for obtaining the equation of the tangent in terms of x and y is to solve the equation

$$f(\xi,\ \eta)=0$$

for η in terms of ξ, there will be n values for η, and substitute these values for η in the equation of the tangent. Assign definite values to ξ and obtain the corresponding values for η in terms of ξ. Since η has n values there will be n tangents, real or imaginary.

A similar discussion applies throughout to the normal.

4. When the locus is of odd degree and symmetrical with respect to the origin linear transformations can be obtained for which the equation is invariant.

Transform the cubic

$$\eta^3-3(\xi^2\eta+\xi^2+\eta^2)=0$$

by the linear transformation

$$\xi=m_1x+c_1y,$$
$$\eta=m_2x+c_2y,$$

the result is

$$x^3(m_2^3-3m_2m_1^2)+x^2y(3m_2^2c_2-3c_2m_1^2-6m_1m_2c_1)-$$
$$3x^2(m_1^2+m_2^2)-6xy(m_1c_1+m_2c_2)+xy^2(3m_2c_2^2-3m_2c_1^2-6c_1c_2m_1)-$$
$$3y^2(c_1^2+c_2^2)+y^3(c_2^3-3c_2c_1^2)=0.$$

The following conditions are sufficient to make the two equations identical :

$$m_1^2+m_2^2=1,$$
$$c_1^2+c_2^2=1,$$
$$m_2^3-3m_2m_1^2=0,$$
$$c_2^3-3c_2c_1^2=1.$$

The solution of these equations gives the two following sets of values either of which will satisfy the condition for invariance :

$$m_1=-\tfrac{1}{2},$$
$$m_2=\tfrac{1}{2}\sqrt{3},$$
$$c_1=-\tfrac{1}{2}\sqrt{3},$$
$$c_2=-\tfrac{1}{2},$$

and

$$m_1=\tfrac{1}{2},$$
$$m_2=\tfrac{1}{2}\sqrt{3},$$
$$c_1=\tfrac{1}{2}\sqrt{3},$$
$$c_2=-\tfrac{1}{2}.$$

The corresponding substitutions are

$$(1)\ \xi=-\tfrac{1}{2}x-\tfrac{1}{2}\sqrt{3}y,$$
$$\eta=\tfrac{1}{2}\sqrt{3}x-\tfrac{1}{2}y,$$

and

$$(2)\ \xi=\tfrac{1}{2}x+\tfrac{1}{2}\sqrt{3}y,$$
$$\eta=\tfrac{1}{2}\sqrt{3}x-\tfrac{1}{2}y.$$

Transformation (1) corresponds to a rotation of the axes through an angle, $\theta=\frac{2\pi}{3}$, and transformation (2) is equivalent to a rotation of $\theta=\frac{\pi}{3}$ and changing y to $-y$. A transformation of this nature is known as a reflection.

From the above it follows that the cubic is invariant to the substitutions

$$\xi=x\cos\frac{2k\pi}{3}-y\sin\frac{2k\pi}{3},$$

$$\eta=x\sin\frac{2k\pi}{3}+y\cos\frac{2k\pi}{3};$$

and

$$\xi=x\cos\frac{2k'\pi}{3}+y\sin\frac{k'\pi}{3},$$

$$\eta=x\sin\frac{k'\pi}{3}+y\cos\frac{k'\pi}{3},$$

where k' is odd.

The transformations for the general case are

$$\xi=x\cos\frac{2k\pi}{n}-y\sin\frac{2k\pi}{n},$$

$$\eta=x\sin\frac{2k\pi}{n}+y\cos\frac{2k\pi}{n};$$

and

$$\xi=x\cos\frac{k'\pi}{n}+y\sin\frac{k'\pi}{n},$$

$$\eta=x\sin\frac{k'\pi}{n}-y\cos\frac{k'\pi}{n},$$

where k' is odd. The first expression for ξ and η represents rotations, and the second represents reflections.

5. Inversion.*

Let k be any point, (ξ, η) on the locus; O the origin; $\rho=\sqrt{(\xi^2+\eta^2)}$ the radius vector of the point k. With O as center and any radius, R, describe a circle. Find a point, (x, y) such that

$$\sqrt{(\xi^2+\eta^2)}.\sqrt{(x^2+y^2)}=R^2.$$

The above conditions give

$$\xi=\frac{xR^2}{x^2+y^2},\quad \eta=\frac{yR^2}{x^2+y^2}.$$

This transformation gives the inverse of the original locus which in the general case assumes the following form :

$$\begin{aligned}\prod_1^n R^2(y-x\alpha_n)=(x^2+y^2)\{&b_1(y-x\alpha_2)\ldots\ldots\ldots\ldots(y-x\alpha_n)+\\ &b_2(y-x\alpha_1)(y-x\alpha_3)\ldots.(y-x\alpha_n)+\\ &\vdots\qquad\quad\vdots\qquad\quad\vdots\qquad\qquad\quad\vdots\\ &b_n(y-x\alpha_1)\ldots\ldots\ldots\ldots(y-x\alpha_{n-1})\}\end{aligned}$$

This inverse is of the $(n+1)$st degree, but if the origin is changed for the general locus it will then have an absolute term and the inverse would then be of the $2n$th degree. In the derivative eqaation the origin is an $(n-1)$-fold point and an n-fold point in the inverse.

Comparison of the general equation and its inverse shows that each is invariant ta the same substitutions if (x^2+y^2) is invariant to the same. For the general linear transformation the conditions for such invariance are

$$m_1^2+m_2^2=1,$$
$$c_1^2+c_2^2=1,$$
$$m_1c_1+m_2c_2=0. \text{ or } \frac{m_1}{m_2}=-\frac{c_2}{c_1}.$$

These are the relations between the coefficients of x and y for a rotation or for a reflection.

In equation (6) change the origin so that the term -2η will disappear and assume $R=1$, the inverse will then be the Lemniscate of Bernoulli.

The inverse of equation (7) is

$$y^3-3x^2y-3(x^2+y^2)^2=0 \text{ (see Fig. 13).}$$

*Williamson's *Differential Calculus*, page 226, seventh edition.

The original locus and the latter touch in their vertices ; both are invariant to the same substitutions ; the inverse to the sides of the fundamental triangle are tangent circles to the inverse at the origin ; the inverse of the tangents at the vertices are tangent circles at the vertices of each curve and pass through the origin ; the two curves have a three-fold symmetry with respect to the lines making angles $k\pi/3$ through the origin.* (See Fig. 13 and Fig. 14.)

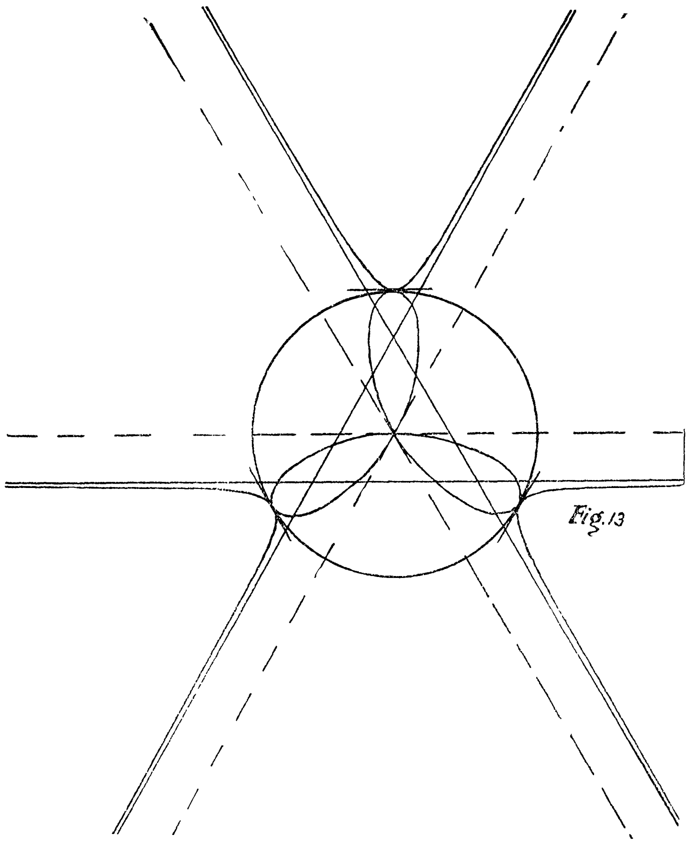

Fig. 13

6. Topological investigations.

The definition and generation of these curves is of such a nature as to render them very suitable for topological investigations. For the sake of brevity these investigations will be confined to n lines which form a polygon in the ordinary sense. The results can be extended to the most general case. The primitive origin will be defined as that origin from which the locus is determined, or through which the variable ray, $y=\lambda x$ passes.

For n lines the locus consists of n branches since the ray goes n times to infinity; $(n-1)$ of these branches pass through the origin ; curves with isolated

*These curves have also been studied by Dr. Arnold Emch of the University of Colorado, in an unpublished article of 1893.

points can be obtained only for positions of the origin within the polygon, the origin being the isolated point; when the origin is external to the polygon $(n-1)$ tangents to the locus can be drawn through it; each tangent is its own inverse; the inverse of each branch cuts the circle of reference in the same two points as does the branch itself; the tangent to each branch through the origin is

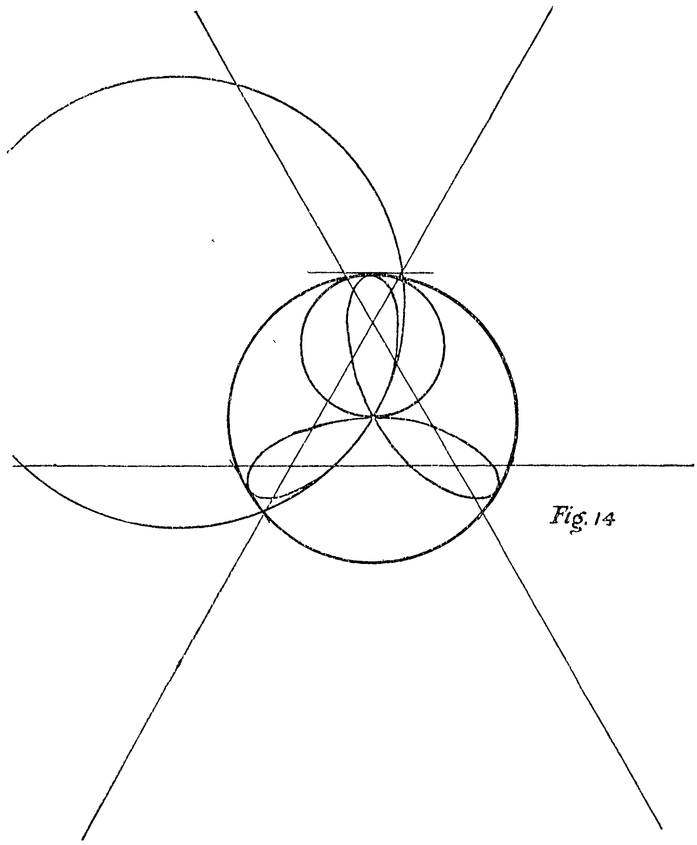

Fig. 14

an asymptote to the inverse of that branch. These statements apply only to the primitive origin and in general when it is not an isolated point. Tangents to the inverse through the origin are parallel to the asymptotes of the given locus; if the origin is not on the locus the inverse consists of n loops passing through this origin which form a closed curve. This fact is of interest because it indicates the continuity through infinity of the original locus in the sense of the theory of transformations. These statements apply to any origin.

The locus of the inverse point for the various branches differs according to the size of the circle of reference as well as for the position of the origin also when the origin is the primitive origin or otherwise.

When a number of branches intersect at the primitive origin and another

origin is then selected the inverse loops then resulting will have a common point of intersection. If the origin is so chosen that one or more of the $(n-1)$ branches through the primitive origin is concave toward it then the inverse of the independent branch will be entirely within the inverse of each and all of the other branches and vice versa if the origin is chosen within the independent branch. If the origin is so that all branches are convex toward it no inverse loop will lie within another.

For two lines one branch of the locus will always pass through the origin; for three lines the origin can be so chosen that, two branches, one branch, or no branch will pass through the origin; for four lines, three, two, or one branch will pass through the origin.

Further investigations in the thesis would seem to indicate that by properly choosing the positions of the n lines and the origin, $(n-1)$, $(n-2)$, ...1, or no branches will pass through the origin according as n is even or odd.

Let n be odd and the intersecting lines form a regular polygon of n sides and the center of gravity be taken as primitive origin. The particular case of the equilateral triangle has been considered.

The locus consists of n branches symmetrically arranged with respect to the origin. Let the circle of reference be tangent to these branches at the vertices. The inverse is a closed curve of n symmetrical loops each of which is tangent to its derivative branch at the vertex and passes through the origin; the locus and its inverse have an n-fold symmetry with respect to the tangents to the inverse through the origin; these tangents are parallel to the sides of the polygon, are n in number and make angles of $k\pi/n$ with each other; the sides of the polygon are asymptotes to the original locus; the inverse of the sides of the polygon are tangent circles to the inverse of the origin; the inverses of the tangents at the vertices are tangent circles at the same points and pass through the origin; the locus and its inverse are invariant to the same substitutions.

If the origin is taken on a line or at the intersection of two lines the problem reduces to that of $(n-1)$ or $(n-2)$ lines. This suggests that if the origin is chosen in the immediate vicinity of such a point the resulting locus will bear a close relation to those of the $(n-1)$st or $(n-2)$nd order. Geometrical comparison of these loci and the algebraic relations between the coefficients are interesting points for investigation. A large amount of topological work has shown up many facts of a less general nature but sufficient to indicate that the problem may be considerably enlarged.

THE SUMMATION OF TWO SERIES.

By G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

As the following series occur in the solution of problem 121, Calculus, the method of summing them, though neither difficult nor tedious, may be of interest to many of our readers.

$$y=1-\frac{n^2x^2}{2!}+\frac{n^2(n^2+8)x^4}{4!}-\frac{n^2(n^4+40n^2+184)x^6}{6!}+\ldots.(A).$$

$$y=\frac{nx}{1!}-\frac{n(n^2+2)x^3}{3!}+\frac{n(n^4+20n^2+24)x^5}{5!}$$

$$-\frac{n(n^6+70n^4+784n^2+720)x^7}{7!}+\ldots(B).$$

The general term of both being

$$a_{r+4}=-\frac{r(r+1)_r+[2(r+2)^2+n^2]a_{r+2}}{(r+3)(r+4)}.$$

From (A), $dy/dx=-n^2x+\dfrac{n^2(n^2+8)x^3}{3!}-\dfrac{n^2(n^4+40n^2+184)x^5}{5!}+\ldots.(1)$.

$$d^2y/dx^2=-n^2+\frac{n^2(n^2+8)x^2}{2!}-\frac{n^2(n^4+40n^2+184)x^4}{4!}+\ldots.(2).$$

n^2 times (A) gives $n^2y=n^2-\dfrac{n^4x^2}{2!}+\dfrac{n^4(n^2+8)x^4}{4!}-\ldots.(3)$.

$2x(1+x^2)$ times (1) gives $2x(1+x^2)(dy/dx)=-2n^2x^2+\dfrac{2n^2(n^2+2)x^4}{3!}-\ldots.(4)$.

$(1+x^2)^2$ times (2) gives

$$(1+x^2)^2(d^2y/dx^2)=-n^2+\frac{n^2(n^2+4)x^2}{2!}-\frac{n^2(n^4+16n^2+16)x^2}{4!}+\ldots.(5).$$

(3)+(4)+(5) gives $(1+x^2)^2(d^2y/dx^2)+2x(1+x^2)(dy/dx)+n^2y=0\ldots$ (6).

From (B), $dy/dx=$

$$n-\frac{n(n^2+2)x^2}{2!}+\frac{n(n^4+20n^2+24)x^4}{4!}-\frac{n(n^6+70n^4+784n^2+720)x^6}{6!}+\ldots.(7).$$

$$d^2y/dx^2=-\frac{n(n^2+2)x}{1!}+\frac{n(n^4+20n^2+24)x^3}{3!}$$

$$-\frac{n(n^6+70n^4+784n^2+720)x^5}{5\,!}\ldots.(8).$$

n^2 times (B) gives

$$n^2y=\frac{n^3x}{1\,!}-\frac{n^3(n^2+2)x^3}{3\,!}+\frac{n^3(n^4+20n^2+24)x^5}{5\,!}-\ldots.(9).$$

$2x(1+x^2)$ times (7) gives

$$2x(1+x^2)(dy/dx)=2nx-\frac{2n^3x^3}{2\,!}+\frac{3n^3(n^2+8)x^5}{4\,!}-\ldots.(10).$$

$(1+x^2)^2(d^2y/dx^2)=$

$$-\frac{n(n^2+2)x}{1\,!}+\frac{n^3(n^2+8)x^3}{3\,!}-\frac{n^3(n^4+30n^2+104)x^5}{5\,!}+\ldots.(11).$$

(9)+(10)+(11) gives $(1+x^2)^2(d^2y/dx^2)+2x(1+x^2)(dy/dx)+n^2y=0$, the same as (6)....(12).

Multiplying (6) and (12) through by $2(dy/dx)$, and integrating, we get $(1+x^2)^2(dy/dx)^2+n^2y^2+C=0$.

When $x=0$, from (A), $y=1$, $dy/dx=0$.

When $x=0$, from (B), $y=0$, $dy/dx=n$.

$\therefore$ In either case, $C=-n^2$. $\therefore$ $(1+x^2)^2(dy/dx)^2=n^2(1-y^2)$.

$$(1+x^2)dy/dx=\pm n\sqrt{(1-y^2)} \text{ or } \frac{dy}{\sqrt{(1-y^2)}}=\pm\frac{ndx}{1+x^2}.$$

$\therefore$ $\sin^{-1}y=n\tan^{-1}x+D$, or $\sin^{-1}y=n\cot^{-1}x+D$.

From (A), when $x=0$, $y=1$.

$\therefore$ $D=\frac{1}{2}\pi$. $\therefore$ $\sin^{-1}y=n\tan^{-1}x+\frac{1}{2}\pi=n\cot^{-1}x+\frac{1}{2}\pi$.

$\therefore$ $y=\sin(n\tan^{-1}x+\frac{1}{2}\pi)=\sin(n\cot^{-1}x+\frac{1}{2}\pi)$.

$\therefore$ $y=\cos(n\tan^{-1}x)=\cos(n\cot^{-1}x)$.

From (B), when $x=0$, $y=0$.

$\therefore$ $D=0$. $\therefore$ $\sin^{-1}y=n\tan^{-1}x=n\cot^{-1}x$.

$\therefore$ $y=\sin(n\tan^{-1}x)=\sin(n\cot^{-1}x)$.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Stock bought m=10% above par, pays p=8% on the investment. What per cent. will it pay if bought at n=10% discount?

II. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; W. P. WEBBER, Mississippi Normal College, Houston, Miss.; F. L. SAWYER, Mitchell, Ontario, Can.; and M. E. GRABER, Heidelberg University, Tiffin, O.

$$p(100+m)\div(100-n)=\frac{p(100+m)}{100-n}\%.$$

But $m=10\%$, $p=8\%$, $n=10\%$.

$$\therefore \frac{8(100+10)}{100-10}=\tfrac{880}{90}=9\tfrac{7}{9}\%.$$

NOTE.—We publish a second solution of this problem because there was some criticism on the solution published in the last number of the MONTHLY. We agreed with Mr. Lawrence in his interpretation of the problem, and for that reason published his solution. We clearly see how the problem may be interpreted in accordance with above solution, and that is the interpretation intended by the proposer. ED. F.

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

According to his contract a professor is to receive \$1800. in cash plus board, etc., for his services during a scholastic year of nine months. This sum is payable in equal installments of \$200. at the end of each scholastic month. The treasurer, however, paid the professor in ten equal installments of \$180. The last two installments were paid Monday and Thursday of the last week in the scholastic year. Regarding money worth 6%, out of how much was the professor defrauded by the wiley treasurer?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$\$20(.04+.035+.03+.025+.02+.015+.01+.005)=\$20\times.18=\$3.60.$$

He paid him \$180 4 days before due.

$$\$180\times.00\tfrac{2}{3}=12 \text{ cents.}\quad \$3.60-\$0.12=\$3.48.$$

149. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A wine cask contains 256 gallons of wine; a certain quantity is drawn off and the cask is filled with water; the same quantity of the mixture is drawn off and the cask is again filled with water and so on for four draughts, when there remain only 81 gallons of wine in the cask. How many gallons of wine are drawn at each of the draughts? [Colburn's *Algebra*.]

I. Solution by JAMES F. LAWRENCE, A. B., Rogers Academy, Rogers, Ark.; S. F. NORRIS, Baltimore City College, Baltimore, Md.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Let $a=256$=the contents of the cask.

x=the amount taken out at each draught, $=64$.

$a-x$=the amount of wine after first drawing.

$\frac{a-x}{a}$=proportion of wine to each gallon of mixture.

$\frac{x(a-x)}{a}$=amount of wine drawn off at second drawing, $=48$.

$\frac{(a-x)^2}{a}$=amount of wine after second drawing.

$\frac{x(a-x)^2}{a^2}$=amount of wine drawn off at third drawing, $=36$.

$\frac{(a-x)^3}{a^2}$=amount of wine after third drawing.

$\frac{x(a-x)^3}{a^2}$=amount of wine drawn off at fourth drawing, $=27$.

$\frac{(a-x)^4}{a^3}$=amount of wine after fourth drawing.

b=amount of wine after fourth drawing.

$$\therefore \frac{(a-x)^4}{a^3}=b. \quad a-x=\sqrt[4]{(a^3b)}. \quad x=a-\sqrt[4]{(a^3b)}=64.$$

Solved in like manner by *C. A. LINDEMANN*, *M. E. GRABER*, and *P. S. BERG*.
Professor Norris sent in a second solution which generalizes the following solution.

II. Solution by D. B. NORTHRUP, Mandana, N. Y., and the PROPOSER.

If one half of the wine is left after the first draught, of course one quarter is left after the second, one eighth after the third, and one sixteenth after the fourth. Hence the number of gallons left after the fourth draught divided by the whole quantity is the fourth power of the proportion left after the first draught. So the fourth root of $\frac{81}{256}=\frac{3}{4}$, the proportion left, and $\frac{1}{4}=64$=number of gallons first drawn, and 48, 36, and 27 the number at each of the other draughts.

ALGEBRA.

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127. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Sum to n terms the series

$$\frac{4}{1.2.3}\cdot\frac{1}{3}+\frac{5}{2.3.4}\cdot\frac{1}{3^2}+\frac{6}{3.4.5}\cdot\frac{1}{3^2}+\cdots$$

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; NELSON L. RORAY, Palmyra, N. Y., and the PROPOSER.

The general term $u_n=\dfrac{n+3}{n(n+1)(n+2)}\cdot\dfrac{1}{3^n}$.

$$u_n=\frac{1}{2n(n+1)}\cdot\frac{1}{3^{n-1}}-\frac{1}{2(n+1)(n+2)}\cdot\frac{1}{3^n}.$$

$$\therefore\ u_1=\frac{1}{2.2.1}-\frac{1}{2.2.3.3},$$

$$u_2=\frac{1}{2.2.3.3}-\frac{1}{2.3.4.3^2},$$

$$u_3=\frac{1}{2.3.4.3^2}-\frac{1}{2.4.5.3^3},$$

. . . .

$$u_n=\frac{1}{2n(n+1).3^{n-1}}-\frac{1}{2(n+1)(n+2).3^n}.$$

Adding, we get for the sum

$$S_n=\tfrac{1}{4}-\frac{1}{2(n+1)(n+2)}\cdot\frac{1}{3^n}.$$

Also solved by *ELMER SCHUYLER*.

128. Proposed by ELMER SCHUYLER, B. Sc., Teacher of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(1+x^3)(1+x^2)(1+x)=30x^2$.

I. Solution by JOHN A. VAN GROOS, B. S., Graduate Student and Assistant in Mathematics, University of Oregon, Eugene, Ore.

$$(1+x^3)(4+x^2)(1+x)=30x^3\ldots.(1).$$

Writing the equation in the form

$$\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=28,$$

by the theory of reciprocal equations, we have

$$v_3+v_2+v_1=28, \text{ where } v_3=z^3-3z,\ v_2=z^2-2,\ v_1=z=x+1/x.$$

Substituting these values, we have

$$z^3+z^2-2z-30=0.$$

By Newton's method of divisors we find that 3 is a root of this equation.

$\therefore\ (z-3)(z^2+4z+10)=0.\quad \therefore\ z=3,\ z=-2+\sqrt{(-6)},\ z=-2-\sqrt{(-6)}$.
Since $z=x+1/x$, we have

$$x=\tfrac{1}{2}(3\pm\sqrt{5}),\ x=\frac{\sqrt{(-6)}-2\pm\sqrt{[-6-4\sqrt{(-6)}]}}{2},$$

$$x=\frac{-\sqrt{(-6)}-2\pm\sqrt{[4\sqrt{(-6)}-6]}}{2},$$

for the roots of the equation.

II. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and P. S. BERG, Larimore, N. D.

Divide through by x^3 and we get

$$(x^2+1/x)(x+1/x)(1+1/x)=30,$$

$$(x^2+1/x+x+1/x^2)(x+1/x)=30.$$

Let $(x+1/x)=y$. $\therefore\ y^3+y^2-2y=30$.
Add $\frac{9}{4}y$ to both members,

$$y^3+y^2+\tfrac{1}{4}y=\tfrac{9}{4}y+30\ldots.(1).$$

y times (1) gives $y^4+y^3+\frac{1}{4}y^2=\frac{9}{4}y^2+30y$. $(y^2+\frac{1}{2}y)^2=\frac{9}{4}y^2+30y$.
Add $10(y^2+\frac{1}{2}y)$ to both members,

$$(y^2+\tfrac{1}{2}y)^2+10(y^2+\tfrac{1}{2}y)=\tfrac{9}{4}y^2+30y+10(y^2+\tfrac{1}{2}y).$$

Completing the square,

$$(y^2+\tfrac{1}{2}y)^2+10(y^2+\tfrac{1}{2}y)+25=\tfrac{49}{4}y^2+35y+25.\quad y^2+\tfrac{1}{2}y+5=\pm(\tfrac{7}{2}y+5).$$

$\therefore\ y^2-3y=0$, or $y^2+4y=-10$.
$\therefore\ y=0,\ 3,\ -2(1\pm\frac{1}{2}\sqrt{-6})$.
$\therefore\ x+1/x=3$, or $x=\frac{1}{2}(3\pm\sqrt{5})$.

$$x+1/x=-2(1+\tfrac{1}{2}\sqrt{-6}),\text{ or } x=-\tfrac{1}{2}(2+\sqrt{-6})\pm\tfrac{1}{2}\sqrt{[4\sqrt{(-6)}-6]}.$$

$$x+1/x=-2(1-\tfrac{1}{2}\sqrt{-6}),\text{ or } x=-\tfrac{1}{2}(2-\sqrt{-6})\pm\tfrac{1}{2}\sqrt{[-4\sqrt{(-6)}-6]}.$$

III. Solution by J. M. BOORMAN, Woodmere, L. I., and the PROPOSER.

Let $1+x^2=zx$. Then

$$[1-x+x^2][1+x^2][1+2x+x^2]=30x^3,\text{ or } [zx-x][zx][zx+2x]=30x^3\ldots.(1).$$

$$z[z-1][z+2]=30\ldots.(2).$$

$\therefore\ z=3$.
Dividing (2) by $z-3$, we have $z^2+4z+10=0$, whence $z=-2\pm\sqrt{-6}$.
Since $x^2+1=zx$, therefore $x^2+1=3x$ or $x^2+1=x[-2\pm\sqrt{-6}]$.

$\therefore\ x=\frac{1}{2}[3\pm\sqrt{5}]$ or $x=\frac{1}{2}\{-2\pm\sqrt{[-6]}\pm\sqrt{[-6\mp\sqrt{(-6)}]}\}$.

Also solved by *JOHN M. ARNOLD*, *GEORGE D. BIRKHOFF*, *M. E. GRABER*, *J. SCHEFFER*, *L. C. WALKER*, and *H. C. WHITAKER*.

GEOMETRY.

THE PYTHAGOREAN THEOREM.

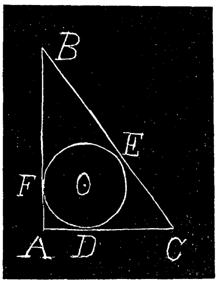
In the supposed "new proof" of the Pythagorean Theorem by Dr. Loomis, which was published last month, there appears a glaring fallacy. A number of our readers have written to us, saying that they do not understand Dr. Loomis' reasoning. The fact is, that the proof as published begs the question. That is, what is desired to be proved, is assumed. Dr. Loomis says that he had a purpose in this. He says he desired to turn the attention of thinkers to the graph submitted. The fallacy entirely escaped our notice until our attention was called to it by several of our contributors. Dr. Loomis' fallacy lies in this: If $4ar+4r^2>$ or $<2bc$, then $a^2+4ar+4r^2>$ or $<b^2+2bc+c^2$, by adding $a^2=b^2+c^2$.

Now we know that $a^2+4ar^2+4r^2=b^2+2bc+c^2$. Hence, if it is assumed that $4ar+4r^2>$ or $<2bc$, the only warranted conclusion is that $a^2<$ or $>b^2+c^2$.

Prof. B. F. Yanney says that such reasoning as employed in the proof given by Dr. Loomis would make $4ar=b^2+c^2$ or $4r^2=b^2+c^2$, or even $r^2=b^2+c^2$.

We publish the following direct proof by Professor Sawyer, which we believe will stand the test of sound reasoning. Similar direct proofs were received from W. H. Carter, D. E. Lehman, Anna L. Benschoten, and Hon. Josiah H. Drummond.

Direct proof by F. L. SAWYER, B. A., Mitchell, Ont.



Connect O with the vertices A, B, and C.

$a+2r=b+c$....(1).

$\therefore\ 2a+2r=a+b+c$.

$\therefore\ 4ar+4r^2=2r(a+b+c)$.

Now the sum of the areas of the triangles AOB, BOC, COA=area of triangle ABC.

$\therefore\ \frac{1}{2}r(c+a+b)=\frac{1}{2}bc$....(2).

$\therefore\ 2r(a+b+c)=2bc$....(3).

$\therefore\ 4ar+4r^2=2bc$ by substituting (1) in (3).

But since $a+2r=b+c$ $\therefore$ $a^2+4ar+4r^2=b^2+c^2+2bc$.

$\therefore\ a^2=b^2+c^2$.

Problem 153, Geometry, is erroneous, it should read as follows:

If P,P', Q,Q', are the extremities of two chords of a conic section passing through the focus, A, and at right angles to each other, show that the sum of the squares of the reciprocals of AP, AP', AQ, and AQ' is constant.

158. Proposed by JOHN MACNIE, A. M., Professor of Latin, University of North Dakota.

Show by a simple diagram that:

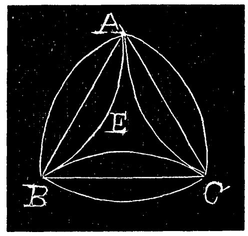
(a) If the angle-sum of an equilateral triangle is constant, that constant is a straight angle.

(b) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(c) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) In the plane equilateral triangle whose sides are equal to AB, the angle sum is constant and equal to a straight angle.



(2) In the spherical equilateral triangle whose sides are equal to AEB, the angle sum is less than a straight angle and approaches a straight angle as the sides approach straight lines.

(3) In the spherical equilateral triangle whose sides are equal to ADB, the angle sum is greater than a straight angle and this sum approaches a straight angle as the sides approach straight lines. In (2) the angle sum must increase and in (3) decrease to approach a straight angle.

Also solved by the *PROPOSER*.

159. Proposed by FRANCIS W. HANNAWALT, Professor of Mathematics and Astronomy in Iowa Wesleyan University. Mt. Pleasant, Iowa.

A man desires to lay out a half-mile race course by using two circles of 150 feet radius and their internal tangents. How far apart shall the circles be placed?

I. Solution by H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Draw the figure to the scale of 1:150. Denote the half distance between centers by x. Then one fourth of the whole path$=\sqrt{[x^2-1]}+\sec^{-1}[-x]=4.4$.

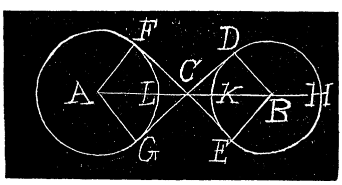
That is; $\sqrt{\{[x+1][x-1]\}}+\sec^{-1}[-x]-4.4=0$.

If $x=1, f(x)=-1.26$; $x=2, f(x)=-0.57$;

$x=3, f(x)=+0.34$; $x=2.63723$.

Required distance$=300x=791.169$ feet.

II. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia. Pa.; and F. ANDEREGG, A. M., Oberlin College, Oberlin, O.



Let $BD=150$ feet$=r$, $\angle DBC=\theta$.

Then $CD=r\tan\theta$, $CB=r\sec\theta$, arc$DH=r(\pi-\theta)$, $LK=2r(\sec\theta-1)$.

Length of course$=4r(\pi-\theta+\tan\theta)=2640$ feet.

$\therefore\ 600(\pi-\theta+\tan\theta)=2640$.

$\pi-\theta+\tan\theta=4.4$. $\tan\theta-\theta=1.2584$.

$\therefore\ \theta=67°\ 43'$ nearly. $AB=2r\sec\theta=791.16$ feet.
$LK=2r(\sec\theta-1)=491.16$ feet.

Also solved by *J. SCHEFFER.*

CALCULUS.

119. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Rectify the Folium of Descartes, the equation of which is $x^3+y^3+3axy=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Substitute $-\dfrac{1}{\sqrt{2}}(x+y)$ and $-\dfrac{1}{\sqrt{2}}(x-y)$ for x and y, respectively, (revolving the axis through 135°) and the equation becomes

$$3y^2(a\sqrt{2}+2x)=x^2(3a\sqrt{2}-2x) \text{ or } 3y^2(a+x\sqrt{2})=x^2(3a-x\sqrt{2}).$$

$$\therefore\ y=\pm\frac{x\sqrt{(3a-x\sqrt{2})}}{\sqrt{[3(a+x\sqrt{2})]}}.$$

$$\frac{dy}{dx}=\frac{3a^2-2x^2}{\sqrt{[3(a+x\sqrt{2})^3(3a-x\sqrt{2})]}}.$$

$$\frac{ds}{dx}=\frac{\sqrt{[2(9a^4+12a^3x\sqrt{2}+12a^2x^2-4x^4)]}}{(a+x\sqrt{2})\sqrt{[3(a+x\sqrt{2})(3a-x\sqrt{2})]}}.$$

Let $a-x\sqrt{2}=2a\sin\theta$.

$$\therefore\ S=-\frac{a}{\sqrt{6}}\int\frac{\sqrt{(13-20\sin\theta+16\sin^3\theta-8\sin^4\theta)}}{1-\sin\theta}d\theta.$$

For the loop, $S=\dfrac{2a}{\sqrt{6}}\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{6}\pi}\frac{\sqrt{(13-20\sin\theta+16\sin^3\theta-8\sin^4\theta)}}{1-\sin\theta}d\theta.$

Also solved by *ELMER SCHUYLER.*

120. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The axis of a paraboloid of revolution coincides with the generating line of a cylinder; the diameter of a cylinder and the latus-rectum of the parabola are each equal to the common altitude, a. Find the surface and volume of each part into which the paraboloid is divided by the cylinder.

Solution by the PROPOSER.

$x^2+y^2=az$ is the equation to the paraboloid.
Volume of paraboloid$=v_1=\frac{1}{2}\pi a^3$.
Surface of paraboloid$=s_1=\frac{1}{6}\pi a^2(5\sqrt{5}-1)$.

$x^2+y^2=ax$ is the equation of the cylinder.

For the surface of the paraboloid within the cylinder,

$$\sqrt{[1+(dy/dx)^2+(dy/dz)^2]}=\frac{\sqrt{a}}{2}\sqrt{\frac{a+4z}{az-x^2}}.$$

$$\therefore\ s=\sqrt{a}\int_0^a\int_z^{\sqrt{az}}\sqrt{\frac{a+4z}{az-x^2}}dzdx$$

$$=\frac{\sqrt{a}}{2}\int_0^a[\pi-2\sin^{-1}\sqrt{(z/a)}]\sqrt{(a+4z)}dz=\frac{\sqrt{a}}{12}\int_0^a\frac{(a+4z)^{\frac{3}{2}}\,dz}{\sqrt{(az-z^2)}}-\frac{\pi}{12}a^2.$$

Let $z=a\sin^2\frac{1}{2}\theta$.

$$\therefore\ s=\frac{a^2}{12}\int_0^\pi[1+4\sin^2\tfrac{1}{2}\theta)]^{\frac{3}{2}}\,d\theta-\frac{\pi}{12}a^2.\quad \text{Let } \theta=\pi-2\varphi.$$

$$\therefore\ s=\frac{a^2}{6}\int_0^{\frac{1}{2}\pi}(5-4\sin^2\varphi)^{\frac{3}{2}}\,d\varphi-\frac{\pi}{12}a^2$$

$$=\frac{a^2}{18}[12E(2/\sqrt{5},\ \tfrac{1}{2}\pi)-F(2/\sqrt{5},\ \tfrac{1}{2}\pi)]-\frac{\pi}{12}a^2.$$

$\therefore$ s and s_1-s are the parts of the surface.

For the volume common to both.

$$v=2\int_0^a\left[\int_z^{\sqrt{az}}\sqrt{(az-x^2)}dx+\int_0^z\sqrt{(ax-x^2)}dx\right]dz$$

$$=\tfrac{1}{8}\int_0^a\left[\pi a^2+\pi az-4a\sqrt{(az-z^2)}-8az\sin^{-1}\sqrt{(z/a)}+2a^2\sin^{-1}\left(\frac{2z-a}{a}\right)\right]dz$$

$$=\tfrac{5}{32}\pi a^3.\quad v_1-v=\tfrac{11}{32}\pi a^3.$$

Also solved by *L. C. WALKER.*

MECHANICS.

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123. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette. Stroud, Gloucestershire, England.

Two equal uniform rods AB, BC are freely hinged at B; C rests on rough horizontal plane, and A is attached to point above it. When C is as far as possible from A for equilibrium, AB, BC make angles α, β, respectively, with the vertical. Find the coefficient of friction between the rod at C and the plane.

governor may also be cited. The whirligig may be considered a practical reversion of the principle exemplified in this problem.

AVERAGE AND PROBABILITY.

NOTE ON THE POND PROBLEM.

It was our purpose to publish a second solution of problem 90, Average and Probability, by a different method, but so far we have been unable to obtain a positive result. We have gone over our investigations a number of times, and have had several of our contributors go over the calculations, and so far we have been unable to find our error. Since we obtain a negative result, there is certainly something wrong with the work, for the problem admits of a definite solution when once the law of distribution has been decided upon. We hope to find time to go over our work again, and should we find our error, we will publish our solution as the method may be of interest to many of our readers.

This problem was originally proposed by Artemas Martin, Ph. D., and published as problem 300, Vol. I., No. 6, page 195, in the *Mathematical Visitor*, edited and published by himself. We infer that this is the source from which Dr. Byerly took it for his problem 21, *Integral Calculus*, second edition, page 211. While no specific reference is made as to the source from which it was taken, yet in his introductory paragraph on the subject of Mean Value and Probability, Dr. Byerly makes mention of the *Mathematical Visitor*.

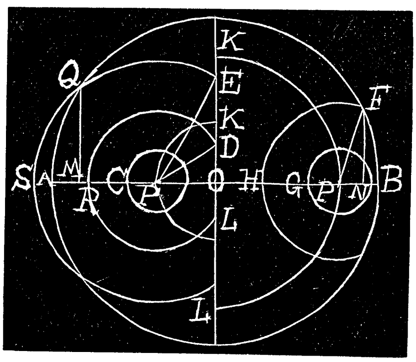
In a letter to us from Dr. Martin, he says that Professor Seitz sent him a solution, giving the same answer as was obtained in the two solutions published in the December number of Vol. VII. ED. F.

106. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Required the average distance between two points in a hemisphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the sphere, P any point in the radius AO or BO. Take the left hemisphere when $OP<\frac{1}{2}r$, the right hemisphere when $OP>\frac{1}{2}r$. Let $OP=x$, PC, PG, PD, PE, $PF=y$.



If the first point is anywhere on the hemisphere with radius x, the second point must be on the surface of the zone of a sphere radius y. The surface of the hemisphere radius x is $2\pi x^2$ when $x<\frac{1}{2}r$. If $y<x$ the second point lies on the surface of a sphere radius y. The area of this surface$=4\pi y^2$. If $y>x$ and $<(r-x)$, the second point is on the area $2\pi PR.OR=$

$2\pi y(y+x)$. If $y>(r-x)$ and $<\sqrt{(r^2+x^2)}$, the second point lies on the area $2\pi OS.OM=(\pi y/x)(r^2+x^2-y^2)$. When $x>\frac{1}{2}r$. The only different area for the second point is when $y>(r-x)$ and $<x$; then the area is $2\pi PH.NH=(\pi y/x)[r^2-(y-x)^2]$. Let Δ be the required average distance.

$$\Delta=\frac{2\pi^2}{(\frac{4}{3}\pi r^3)^2}\int_0^{\frac{1}{2}r} x^2dx\left[\int_0^x 4y^3dy+\int_x^{r-x} 2y^2(y+x)dy+\int_{r-x}^{\sqrt{(r^2+x^2)}}(y^2/x)(r^2+x^2-y^2)dy\right]$$

$$+\frac{2\pi^2}{(\frac{4}{3}\pi r^3)^2}\int_{\frac{1}{2}r}^r x^2dx\left[\int_0^{r-x}4y^3dy+\int_{r-x}^x(y^2/x)[r^2-(y-x)^2]dy\right.$$

$$\left.+\int_x^{\sqrt{(r^2+x^2)}}(y^2/x)(r^2+x^2-y^2)dy\right]$$

$$=\frac{3}{20r^6}\int_0^r [15r^4x^2-20r^3x^3+10r^2x^4-6x^6-4r^5x+4x(r^2+x^2)^{\frac{5}{2}}]dx$$

$$=\tfrac{3}{70}(16\sqrt{2}-5)r.$$

If one point is taken in each hemisphere we get

$$\Delta_1=\frac{2\pi^2}{(\frac{4}{3}\pi r^3)^2}\int_0^r x^2dx\left[\int_x^{\sqrt{(r^2+x^2)}}2y^2(y-x)dy+\int_{\sqrt{(r^2+x^2)}}^{r+x}(y^2/x)[r^2-(y-x)^2]dy\right]$$

$$=\frac{3}{20r^6}\int_0^r[4r^5x+15r^4x^2+20r^3x^3+10r^2x^4+4x^6-4x(r^2+x^2)^{\frac{5}{2}}]dx$$

$$=\tfrac{3}{70}(53-16\sqrt{2})r.$$

If both points are taken anywhere in the sphere we get

$$\Delta_2=\frac{4\pi^2}{(\frac{4}{3}\pi r^3)^2}\int_0^r x^2dx\left[\int_0^{r-x}4y^3dy+\int_{r-x}^{r+x}(y^2/x)[r^2-(y-x)^2]dy\right]$$

$$=\frac{3}{20r^6}\int_0^r(15r^4+10r^2x^2-x^4)x^2dx=\frac{36r}{35}.$$

$$\Delta=2\Delta_2-\Delta_1=\tfrac{3}{70}(16\sqrt{2}-5)r.$$

These methods greatly simplify the tedium of integration.

MISCELLANEOUS.

98. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.**

A particle describes an ellipse under an attraction always directed to the vertex; to determine the law of the attraction.

Solution by the PROPOSER.

$r(a^2\sin^2\theta+b^2\cos^2\theta)=2ab^2\cos\theta$ or $u=\dfrac{a^2\sin^2\theta+b^2\cos^2\theta}{2ab^2\cos\theta}$ is polar equation to the ellipse with vertex as pole.

$F=h^2u^2\left(\dfrac{d^2u}{d\theta^2}+u\right)$ is the force required, h being an undetermined constant.

$$\frac{d^2u}{d\theta^2}=\frac{(a^2-b^2)\cos^4\theta+a^2\sin^2\theta+a^2}{2ab^2\cos^3\theta};\quad \frac{d^2u}{d\theta^2}+u=\frac{a}{b^2\cos^3\theta}.$$

$$\therefore\ F=\frac{ah^2u^2}{b^2\cos^3\theta}=\frac{ah^2(a^2\sin^2\theta+b^2\cos^2\theta)^2}{4a^2b^6\cos^5\theta},$$

$$\therefore\ F=\frac{a^3h^2}{4b^6\cos^5\theta}-\frac{h^2a(a^2-b^2)}{2b^6\cos^3\theta}+\frac{(a^2-b^2)^2h^2}{4ab^6\cos\theta}.$$

Also solved by *J. SCHEFFER.*

99. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Through the zenith of an observer at the sea-coast in north latitude $\varphi=40°$, a "cat's-tail cloud," height $h=10$ miles, extends northeast until it touches the horizon. How far from the observer is the *advance-end* of the cloud? What is the length of the cloud measured from the end specified to the observer's zenith?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Regarding the earth as a perfect sphere of radius 3963 miles, and disregarding the observer's height, the distance and length required will be the same for any latitude.

Let C be the center of the earth, O the position of the observer, Z his zenith, and BOA the diameter of the visible horizon.

$OA=\sqrt{(CA^2-CO^2)}=\sqrt{[(3973)^2-(3963)^2]}=281.71$ miles, the distance of advance end from observer.

$\cos OCA=\frac{3963}{3973}=.997483.\quad \angle OCA=4°\ 4'.$

$\therefore\ ZA=(4\frac{1}{15}\pi\times7946)/360=281.99$ miles, length of cloud.

100. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Determine the maximum value of $(\varphi-\varphi')$, if given electric currents C and C' produce deflections φ and φ' in a tangent galvanometer, so that $\tan\varphi/\tan\varphi'=C/C'$.

Solution by GEORGE R. DEAN, Professor of Mathematics, Missouri State School of Mines and Metallurgy, Rolla, Mo.

We have $C'\tan\varphi=C\tan\varphi'$, $u=\varphi-\varphi'$.

Differentiating, we must have for a maximum or minimum, $C'\sec^2\varphi d\varphi=C\sec^2\varphi' d\varphi'$. $d\varphi=d\varphi'$.

Eliminating the differentials, $C'\sec^2\varphi=C\sec^2\varphi'$.

Solving this and the given relation for φ, we find

$$\tan^2\varphi=\frac{C^2-CC'}{C'^2-CC'}=-\frac{C}{C'}.$$

But $\dfrac{-C}{C'}=-\dfrac{\tan\varphi}{\tan\varphi'}$. Hence $\tan\varphi\tan\varphi'=-1$.

This shows that $\varphi-\varphi'=90°$.

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

145. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve the equations:

$$x+y+z+u+w=1,$$
$$ax+by+cz+du+ew=h,$$
$$a^2x+b^2y+c^2z+d^2u+e^2w=h^2,$$
$$a^3x+b^3y+c^3z+d^3u+e^3w=h^3,$$
$$a^4x+b^4y+c^4z+d^4u+e^4w=h^4.$$

146. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Solve by a short original method, if possible:

$$x/a+y/b+c/z=P\ldots.(1),$$
$$x/a+b/y+z/c=Q\ldots.(2),$$
$$a/x+y/b+z/c=R\ldots.(3).$$

GEOMETRY.

180. Proposed by R. TUCKER, M. A.

ABC is a triangle; A', B', C' are the images of A, B, C with respect to BC, CA, AB. The circum-circle ABC cuts $A'BC$ (say) in K (on $A'B$), M (on $A'C$), and AK, AM, AA' cut BC in P, R, Q, respectively. Prove that (1) the orthocenters of the associated triangles lie on circle ABC; (2) triangle AKM has its sides parallel to and equal twice the sides of the pedal triangle of ABC, and is also equal triangle formed by the above-named orthocenters; (3) $CP.a=b^2$,

$BR.a=c^2$, $AP.a=AR.a=bc$, $BP.a=a^2-b^2$, $CR.a=a^2-c^2$, *i. e.*, $PR.a=2bc\cos A$; (4) hence BA touches circle ARC, which contains a Brocard-point of ABC; similarly for CA and circle APB; (5) $BR.CR'$, $AR''=abc=CP.BP'$, AP'' (where R', R'', P', P'' correspond to R, P, on CA, AB, respectively); K, K' are the Brocard constants ($k=a^2+b^2+c^2$) of ABC, $A'B'C'$: then $K'-K=\triangle^2/R^2$.

181. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the extremities of the latera recta of all ellipses having a given major axis $2a$ lie on the parabola $x^2=-a(y-a)$.

CALCULUS.

142. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the differential equation,

$$(a-x)\frac{dz}{dx}+(b-y)\frac{dz}{dy}=c-z.$$

143. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Find the area of greatest ellipse that can be inscribed in a given semicircle.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume of the sphere, $x^2+y^2+z^2=2az$, (a) within the paraboloid $z=Ax^2+By^2$; (b) within the cone $z^2=Ax^2+By^2$.

MECHANICS.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $pv=Rt-b/tv$ be the equation for CO_2 gas, find the total, external and internal work done in compressing the gas from 102 to 136 atmospheres at a constant temperature 17°C and constant volume, $R=63.23$, $b=481600$ for CO_2.

135. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

What force acting *at an inclination* ω with a horizontal line on the center of a wheel of given weight will *roll* the wheel over an immovable cylindric log whose diameter is $(1/m)$th that of the wheel?

DIOPHANTINE ANALYSIS.

91. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

There are two unequal squara numbers the sum of whose sum, difference, product, and quotient, is a square. Find the two numbers.

92. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

(*a*) Find the least three integral numbers such that the difference of every two of them shall be a square number; (*b*) Find the least three square numbers such that the difference of every two of them shall be a square number.

AVERAGE AND PROBABILITY.

119. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

Two players throw three dice, the object being to throw at one cast a tres, a deux, and an ace. They continue throwing in succession until one of the players wins. What advantage has the first player?

120. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

If a given ellipse be placed at random on another equal ellipse, find the chance that the center of the first will fall on the second.

NOTES.

The Royal Society of London has awarded its Copley Medal to Prof. J. W. Gibbs, of Yale, for his contributions in Mathematical Physics.

We have in our possession some excellent articles for next year. In the January number will appear an article *Determination of all the Groups of Order* 168, by Dr. G. A. Miller, of Leland Stanford Jr. University.

According to a new post-office ruling at Washington, D. C., magazines are not allowed to be sent to subscribers longer than the time for which the subscription has been paid. In view of this ruling, it is necessary that our subscribers send in their subscriptions at once. Notices have been sent to all. Please respond to same immediately.

The New Geometry Waking the Sleep of Two Thousand Years is the title of an article which the new monthly, *Everybody's Magazine*, has secured from Dr. Halsted, to appear in the February issue.

If, by the management of a phenominally successful journal, this article is expected to seize and hold the attention of its readers, what *must* be its interest for actual mathematicians? We think that it is no exaggeration to say that Dr. Halsted's Supplementary Report on Non-Euclidean Geometry at the Denver Meeting of the Association for the Advancement of Science aroused greater general interest than any other paper presented at the meeting. A like interest will certainly be taken in the forthcoming article. Our readers should be sure to secure a copy of the magazine containing this article.